BIRZEIT UNIVERSITY

Faculty of Science

Master Program of Mathematics

DYNAMICS AND STABILITY OF SOME SYSTEMS OF DIFFERENCE EQUATIONS

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Palestine

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This thesis was successfully defended on June 25th, 2019. And approved by the committee members:

1. Dr. Marwan Aloqeili (Head of Committee) ....

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Dedication

To my father who I would never be the same person I am now without him. For his endless love, infinite patience and support through all the times I've been down.

To my family for their support, to my friends for their love. To all who have ever taught me, believed in me and gave me a chance.

I gratefully dedicate this work to each one of you.

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DECLARATION

I certify that this thesis, which was submitted to the Department of Mathematics in Birzeit University as a partial fulfillment of the Master's degree requirements, is of my own research except where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any university or institution.

Sima Abualrub ....

June 25th, 2019

Abstract

In this research, we investigate semi-cycles, boundedness, persistence of positive solutions, and global asymptotic stability of the unique positive equilibrium of two different systems of two nonlinear difference equations. The first system is:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots.$$

where A, B are positive real numbers, the initial conditions $x_i, y_i \in (0, \infty)$ for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$. The second system is:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \ldots.$$

with parameter $A \in (0, \infty)$, and x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$.

ملخص

في هذه الرسالة، نقوم بدراسة الخصائص الديناميكية للحلول الموجبة لنظامين من معادلتي فرق من الدرجات العليا. النظام الأول هو:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots.$$

حيث أن A, B هما عددان حقيقيان موجبان، و $x_i, y_i \in (0, \infty)$ لكل $i = -k, -k+1, ..., 0$ ، حيث k عدد صحيح موجب. بينما النظام الثاني هو:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \ldots.$$

حيث أن $A \in (0, \infty)$ ، و x_i, y_i هي أعداد حقيقية موجبة لكل $i = -k, -k+1, ..., 0$ و $k \in \mathbb{Z}^+$.

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Preliminaries

1.1 Introduction

Discrete dynamical systems and difference equations have captured the interest of the researchers in the last few years, especially these equations which arise in mathematical models that describe problems in physics, biology, economics and engineering. Studying the dynamical behavior of difference equations and systems is not only of interest in their own right, but the results can help to develop the theory of difference equations. Difference equations might sometimes have simple forms, however, it is crucially hard to fully understand the behavior of their solutions.

Recently, nonlinear difference equations and systems are of wide interest [1–6, 9–25]. Particularly, in 1998, Papaschinopoulos and Schinas [16] studied the oscillatory behavior, periodicity and boundedness of the solutions of the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \dots \tag{1.1}$$

where $A > 0$ and p, q are positive integers. They proved that any positive solution of (1.1) oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$, and if $A > 0$ and at least one of p, q is an odd number (respectively, $A > 1$ and p, q are both even numbers), then any positive solution of (1.1) is bounded. Moreover, they proved that if $A > 1$, then the unique positive equilibrium of system (1.1) is globally asymptotically stable. Moreover, they considered system (1.1) in the case that $A = 0$ and $p = q = 1$, and found that every solution of system (1.1) in this case is periodic of period 6.

After that, in 2000, Papaschinopoulos and Schinas [17] investigated the system:

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0, 1, \dots \tag{1.2}$$

where A is a positive constant and x_{-1}, x_0, y_{-1}, y_0 are positive numbers. They proved that the positive solution of system (1.2) oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$. Moreover, they proved that system (1.2) has periodic solutions of period two if $A = 1$, and that any positive solution of system (1.2) tends to the equilibrium as $n \to \infty$. Furthermore, they showed that if $0 < A < 1$, then system (1.2) has unbounded solutions. If $A = 1$, then every positive solution of (1.2) tends to a periodic solution of period two, and if $A > 1$ then the positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ of (1.2) is globally asymptotically stable.

Whereas Papaschinopoulos and Papadopoulos [15] studied, in 2002, the existence of positive solutions of the equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-m}}, \quad n = 0, 1, \dots \tag{1.3}$$

they found that there exist bounded and unbounded solutions of (1.3). They also introduced the following system of difference equations:

$$x_{n+1} = A + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots \tag{1.4}$$

where $m \in \{1, 2, \dots\}$, and $x_{-m}, x_{-m+1}, \dots, x_0, y_{-m}, y_{-m+1}, \dots, y_0$ are positive constants and A, B are positive real numbers. They proved that if $A > 1$ and $B > 1$, then the solution of (1.4) is bounded and persists, and there will be a unique positive equilibrium $(\bar{x}, \bar{y})$ of system (1.4), and that every positive solution of (1.4) tends to that unique positive equilibrium as $n \to \infty$. They could also found unbounded solutions when $0 < A < 1$ or $0 < B < 1$.

In 2004, Camouzis and Papaschinopoulos [2] studied the boundedness and persistence of the positive solutions of the following system:

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots \tag{1.5}$$

where x_i, y_i are positive numbers for $i = -m, -m+1, \dots, 0$ and m is a positive integer. Furthermore, they proved that (1.5) has an infinite number of positive equilibrium solutions and that every positive solution converges to a positive equilibrium solution $(\bar{x}, \bar{y}) = (2, 2)$ as $n \to \infty$.

In 2007, Y. Zhang et al. [25] introduced the system:

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad n = 0, 1, \dots \tag{1.6}$$

where the parameter A and the initial conditions x_i, y_i are positive real numbers for $i = -m, -m+1, \ldots, 0$, and m is a positive integer. Zhang et al. proved that the unique positive equilibrium of (1.6) is globally asymptotically stable for $A > 1$, and the positive solution of system (1.6) is bounded and persists when $A \geq 1$, they also found unbounded solutions of system (1.6) when $0 < A < 1$, and showed that for $A = 1$, if m is odd then any positive solution of (1.6) with prime period two is of the form

$$\ldots, (a, a), (\frac{a}{a-1}, \frac{a}{a-1}), (a, a), (\frac{a}{a-1}, \frac{a}{a-1}), \ldots$$

where $1 < a \neq 2$, however, if m is even then any positive solution of (1.6) with prime period two takes the form

$$\ldots, (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), \ldots$$

where $1 < a \neq 2$.

While Q. Zhang, Yang, and Liu [24] in 2013 investigated the boundedness, persistence of positive solutions and global asymptotic stability of the positive equilibrium of the system:

$$x_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad y_{n+1} = B + \frac{y_{n-m}}{x_n}, \quad n = 0, 1, \ldots \tag{1.7}$$

where $A, B, x_i, y_i \in (0, \infty)$ for $i = -m, -m+1, \ldots, 0$ and $m \in \mathbb{Z}^+$. They found unbounded solutions for system (1.7) when A and B are less than one, and proved that when $A \geq 1$ and $B \geq 1$ the positive solution of system (1.7) is bounded and persists, and when $A > 1$ and $A > 1$ the positive equilibrium point $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ is globally asymptotically stable.

In 2014, Q. Zhang et al. [23] investigated the global asymptotic behavior of the system of the following two rational difference equations:

$$x_{n+1} = A + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}, \quad y_{n+1} = B + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots \tag{1.8}$$

where A, B, x_i, y_i are positive real numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$. More precisely, Zhang et al. proved that if $A > \frac{1}{k}$ and $B > \frac{1}{k}$, then every positive solution of system (1.8) is bounded and persists. Moreover, they proved that every positive solution converges to the positive equilibrium $(\bar{x}, \bar{y})$ as $n \to \infty$.

Finally, D. Zhang et al. [22] introduced the system

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots \tag{1.9}$$

with parameter $A > 0$, and the initial conditions x_i, y_i are arbitrary positive real numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$. They studied the asymptotic behavior of positive solutions of the system in the cases $0 < A < 1$, $A = 1$ and $A > 1$. When $0 < A < 1$, they could find unbounded solutions of system (1.9), and when $A = 1$ they proved that system (1.9) can have two periodic solutions, and any positive solution is bounded and persists. They also proved that the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is a global attractor when $A > 1$. Later in 2018, Gumus [10] investigated the semi-cycles of the positive solutions for the same system. They also proved that if $A > 1$ then the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is globally asymptotically stable.

Other related difference equations and systems can be found in references [1, 3–6, 9, 11–14, 18–21]. More details about the theory of difference equations are provided in [7, 8].

Motivated by all the systems we previously mentioned, we introduce in Chapter 2 the system

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$. In Chapter 3, we introduce the system

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \ldots$$

with parameter $A > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$. As far as we know, no work has been reported in the literature on the dynamics of these two system.

In Chapter 2, we study the semi-cycles of the positive solutions of system (2.1), we also find unbounded solutions of the same system when $0 < A < 1$ and $0 < B < 1$, we prove that the positive solutions of system (2.1) are bounded and persist for $A \geq 1$ and $B \geq 1$. Finally, we show that if $A > 1$ and $B > 1$ then the unique positive equilibrium of system (2.1) is globally asymptotically stable. Moreover, in Chapter 3, we investigate system (3.1) via semi-cycle analysis method, and then we assume some conditions to get unbounded solutions for this system. We also prove that if $A \geq 1$ then every positive solution of system (3.1) is bounded, and if $A = 1$ then the system can have a two periodic solution. Then, we show that the positive equilibrium of system (3.1) is globally asymptotically stable when $A > 1$.

We conclude each chapter of these two chapters by numerical examples that supports our analytical results.

1.2 Basic Definitions and Results

In this section, we provide basic definitions and results that we're about to use in the following chapters. Consider the $2(k+1)$-dimensional dynamical system of the following form:

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k}) \tag{1.10}$$
$$y_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k})$$
$$n = 0, 1, ...$$

where f, g are continuously differentiable real valued functions.

Definition 1.1 (Equilibrium Point)**.** A point $(\bar{x}, \bar{y})$ is said to be an equilibrium point of system (1.10) if

$$\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x}, \bar{y}, \bar{y}, ..., \bar{y}) \tag{1.11}$$
$$\text{and} \quad \bar{y} = g(\bar{x}, \bar{x}, ..., \bar{x}, \bar{y}, \bar{y}, ..., \bar{y})$$

Definition 1.2 (Stable, Unstable, Attracting, Asymptotically Stable and Globally Asymptotically Stable Equilibrium Point)**.** If $(\bar{x}, \bar{y})$ is an equilibrium point of (1.10), then

1. $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-k, -k+1, ..., 0\}$ if $\| \sum_{i=-k}^{0}(x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$ implies that for all $n > 0$, $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$, where $\|.\|$ is usual Euclidian norm in $\mathbb{R}^2$. Otherwise, $(\bar{x}, \bar{y})$ is called unstable.

2. An equilibrium point $(\bar{x}, \bar{y})$ is called attracting if there exists $\eta > 0$ such that

$$\| \sum_{i=-k}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta \text{ implies } \lim_{n\to\infty} (x_n, y_n) = (\bar{x}, \bar{y}) \quad (1.12)$$

3. $(\bar{x}, \bar{y})$ is called a global attractor if in 2, $\eta = \infty$.

4. An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotically stable if it is both stable and attracting, and it is said to be globally asymptotically stable if it is both stable and global attractor.

Definition 1.3 (Positive Solution)**.** A pair of sequences of positive real numbers $\{x_n, y_n\}_{n=-k}^{\infty}$ that satisfies (1.10) is a positive solution of (1.10).

Definition 1.4 (Equilibrium Solution)**.** If a positive solution of (1.10) is a pair of constants $(\bar{x}, \bar{y})$, then the solution is the equilibrium solution.

Definition 1.5 (Periodic Solution)**.** A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.10) is said to be periodic if there exists a positive integer m, such that for all $n \geq -k$, $(x_n, y_n) = (x_{n+m}, y_{n+m})$. m is called the period of the solution.

Definition 1.6 (Eventually Periodic Solution)**.** A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.10) is said to be eventually periodic if there exist an integer $l > -k$ and a positive integer m, such that $(x_{n+l}, y_{n+l}) = (x_{n+l+m}, y_{n+l+m})$ for all $n = 0, 1, \ldots$ where m is the period of the solution.

Definition 1.7 (Bounded Solution)**.** A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.10) is bounded and persists if there exist positive real numbers P_1, Q_1, P_2 and Q_2 such that $P_1 \leq x_n \leq Q_1$ and $P_2 \leq y_n \leq Q_2$ for $n \geq -k$.

Definition 1.8 (Increasing and Decreasing Solution)**.** A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.10) is said to be increasing (respectively decreasing) if $n > m$, then $x_n > x_m$ and $y_n > y_m$ (respectively $x_n < x_m$ and $y_n < y_m$) for all $n \geq 1$ and $m \geq 1$.

Definition 1.9 (Positive and Negative Semi-cycles)**.** A string of consecutive terms $\{x_t, \ldots, x_r\}$ (respectively $\{y_t, \ldots, y_r\}$), $t \geq -k$, and $r \leq \infty$ is said to be a positive semi-cycle if $x_i \geq \bar{x}$ (respectively $y_i \geq \bar{y}$), $i \in \{t, \ldots, r\}$, $x_{t-1} < \bar{x}$ (respectively $y_{t-1} < \bar{y}$), and $x_{r+1} < \bar{x}$ ($y_{r+1} < \bar{y}$).

A string of consecutive terms $\{x_t, \ldots, x_r\}$ (respectively $\{y_t, ..., y_r\}$), $t \geq -k$, and $r \leq \infty$ is said to be a negative semi-cycle if $x_i < \bar{x}$ (respectively $y_i < \bar{y}$), $i \in \{t, \ldots, r\}$, $x_{t-1} \geq \bar{x}$ (respectively $y_{t-1} \geq \bar{y}$), and $x_{r+1} \geq \bar{x}$ ($y_{r+1} \geq \bar{y}$).

A string of sequential terms $\{(x_t, y_t), \ldots, (x_r, y_r)\}$, $t \geq -k$, and $r \leq \infty$ is said to be a positive semi-cycle (respectively negative semi-cycle) if both $\{x_t, \ldots, x_r\}$ and $\{y_t, \ldots, y_r\}$ are positive semi-cycles (respectively negative semi-cycles).

Finally, a string of sequential terms $\{(x_t, y_t), \ldots, (x_r, y_r)\}$, and $t \geq -k$, $r \leq \infty$ is said to be a positive semi-cycle (respectively negative semi-cycle) with respect to x_n and negative semi-cycle(respectively positive semi-cycle) with respect to y_n if $\{x_t, \ldots, x_r\}$ is a positive semi-cycle (respectively negative semi-cycle) and $\{y_t, \ldots, y_r\}$ is a negative semi-cycle (respectively positive semi-cycle).

The first semi-cycle of a solution of (1.10) starts with the term (x_{-k}, y_{-k}), and it's positive (respectively negative) if $x_{-k} \geq \bar{x}$ and $y_{-k} \geq \bar{y}$ (respectively $x_{-k} < \bar{x}$ and $y_{-k} < \bar{y}$).

Definition 1.10 (Nonoscillatory Solution)**.** A function x_n (respectively y_n) is called nonoscillatory about $\bar{x}$ (respectively $\bar{y}$) if there exists $N \geq -k$ such that $x_n \geq \bar{x}$ (respectively $y_n \geq \bar{y}$) or $x_n < \bar{x}$ (respectively $y_n < \bar{y}$) for all $n \geq N$.

We say that a solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of system (1.10) is a nonoscillatory solution about $(\bar{x}, \bar{y})$ if x_n is nonoscillatory about $\bar{x}$ and y_n is nonoscillatory about $\bar{y}$. However, a solution $\{x_n, y_n\}_{n=-k}^{\infty}$ is called oscillatory if it is not nonoscillatory.

Definition 1.11 (Nonoscillatory Positive and Nonoscillatory negative Solutions)**.** A solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of system (1.10) is a nonoscillatory positive (respectively negative) solution about $(\bar{x}, \bar{y})$ if there exists $N \geq -k$ such that $x_n \geq \bar{x}$ and $y_n \geq \bar{y}$ (respectively $x_n < \bar{x}$ and $y_n < \bar{y}$) for all $n \geq N$.

Definition 1.12 (Linearized Form of (1.10))**.** Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1.10) where f, g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (1.10) about the equilibrium point $(\bar{x}, \bar{y})$ has the form:

$$X_{n+1} = JX_n$$

where $X_n = (x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k})^T$ and J is a Jacobian matrix of system (1.10) about the equilibrium point $(\bar{x}, \bar{y})$.

Theorem 1.2.1. *For the linearized system* $X_{n+1} = JX_n,\ n = 0, 1, ...$ *of (1.10). If all eigenvalues of the Jacobian matrix J about* $(\bar{x}, \bar{y})$ *lie inside the* *open unit disk* $|\lambda| < 1$*, then* $(\bar{x}, \bar{y})$ *is locally asymptotically stable. If one of them has a modulus greater than one, then* $(\bar{x}, \bar{y})$ *is unstable.*

Definition 1.13 (Limit Superior and Limit Inferior)**.** Let $\{x_n\}$ be a sequence of real numbers. The limit superior of $\{x_n\}$, denoted by $\limsup\{x_n\}$, is defined by

$$\limsup\{x_n\} = \lim_{n\to\infty} [\sup \{x_m; m \geq n\}] = \inf_{n\geq 0} [\sup \{x_m; m \geq n\}]$$

The limit inferior of $\{x_n\}$, denoted by $\liminf\{x_n\}$, is defined by

$$\liminf\{x_n\} = \lim_{n\to\infty} [\inf \{x_m; m \geq n\}] = \sup_{n\geq 0} [\inf \{x_m; m \geq n\}]$$

Definition 1.14 (Spectral Radius)**.** Let M be any real $n \times n$ matrix, and assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of M. Then the spectral radius of M, denoted by $\rho(M)$, is given by:

$$\rho(M) = \max_{1\leq i\leq n} \{|\lambda_i|\}$$

Theorem 1.2.2. *Let* $\|.\|$ *be any matrix norm defined on the set of all real* $n \times n$ *matrices* ($\mathcal{M}_n$)*. Then for any matrix* $M \in \mathcal{M}_n$

$$\rho(M) \leq \|A\|$$

Definition 1.15 (Infinite Norm of a Matrix)**.** Let M ba any matrix in $\mathcal{M}_n$. The infinite norm of M, denoted by $\|M\|_\infty$, is given by:

$$\|M\|_\infty = \max_{1\leq r\leq n} \sum_{c=1}^{n} |m_{r,c}|$$

Chapter 2

Dynamics of the System $x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}$

In this chapter, we introduce the following dynamical system:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots \tag{2.1}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$. We study the dynamical behavior of this system in the cases: when $0 < A < 1$ and $0 < A < 1$, and when $A > 1$ and $B > 1$, we also investigate the behavior of the positive solutions of (2.1) using the semi-cycle analysis method. Finally, we give some numerical examples that illustrate the results in this chapter.

System (2.1) has the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$. Since $f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ implies $\bar{x} = A + \frac{\bar{y}}{\bar{y}} = A + 1$, and $\bar{y} = B + \frac{\bar{x}}{\bar{x}} = B + 1$, so $(\bar{x}, \bar{y}) = (A+1, B+1)$.

There are two cases to be considered:

Case 1: if $A = B$, then system (2.1) turns into the symmetrical system (1.9)

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, y_{n+1} = A + \frac{x_{n-k}}{x_n}, n = 0, 1, \ldots.$$

with parameter $A > 0$ and x_i, y_i are positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$, which was studied in [10, 22].

Case 2: when $A \neq B$. This is what we're studying.

2.1 Semi-cycle Analysis

In this section, we examine the behavior of positive solutions of system (2.1) via semi-cycle analysis method.

Theorem 2.1.1. *Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution of system (2.1). Then, either this solution consists of a single semi-cycle or it oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ with semi-cycles having at most k terms.*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is a solution of system (2.1) which has at least two semi-cycles. Then one of these semi-cycles is positive and the other is negative, that is, there exists $n_0 > -k$ such that

$$x_{n_0} < 1 + A \leq x_{n_0+1} \quad \text{and} \quad y_{n_0} < 1 + B \leq y_{n_0+1}$$

or

$$x_{n_0} \geq 1 + A > x_{n_0+1} \quad \text{and} \quad y_{n_0} \geq 1 + B > y_{n_0+1}$$

Case 1: if $x_{n_0} < 1 + A \leq x_{n_0+1}$ and $y_{n_0} < 1 + B \leq y_{n_0+1}$, assume that the positive semi-cycle which starts with (x_{n_0+1}, y_{n_0+1}) has k terms. Then

$$x_{n_0} < 1 + A \leq x_{n_0+1}, \ldots, x_{n_0+k} \text{ implies } \frac{x_{n_0}}{x_{n_0+k}} < 1$$

and

$$y_{n_0} < 1 + B \leq y_{n_0+1}, \ldots, y_{n_0+k} \text{ implies } \frac{y_{n_0}}{y_{n_0+k}} < 1$$

for

$$x_{n_0+k+1} = A + \frac{y_{n_0}}{y_{n_0+k}} < A + 1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0}}{x_{n_0+k}} < B + 1$$

so the semi-cycle has at most k terms.

Case 2: if $x_{n_0} \geq 1 + A > x_{n_0+1}$ and $y_{n_0} \geq 1 + B > y_{n_0+1}$, assume that the negative semi-cycle that starts with (x_{n_0+1}, y_{n_0+1}) has k terms. Then

$$x_{n_0} \geq 1 + A > x_{n_0+1}, \ldots, x_{n_0+k} \text{ implies } \frac{x_{n_0}}{x_{n_0+k}} > 1$$

and

$$y_{n_0} \geq 1 + B > y_{n_0+1}, \ldots, y_{n_0+k} \text{ implies } \frac{y_{n_0}}{y_{n_0+k}} > 1$$

for

$$x_{n_0+k+1} = A + \frac{y_{n_0}}{y_{n_0+k}} > A + 1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0}}{x_{n_0+k}} > B + 1$$

so the semi-cycle has at most k terms. Hence, the result follows. □

Theorem 2.1.2. *Let k be an odd integer and $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution of* *system (2.1) which has $k-1$ sequential semi-cycles of length one. Then, every semi-cycle after this point is of length one.*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is a solution to system (2.1) which has $k-1$ sequential semi-cycles of length one and k is odd. Then there exists $n_0 \geq -k$ such that either

$$x_{n_0}, x_{n_0+2}, ..., x_{n_0+k-1} < 1 + A \leq x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k}{}^{1}$$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k-1} < 1 + B \leq y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k}$$

or

$$x_{n_0}, x_{n_0+2}, ..., x_{n_0+k-1} \geq 1 + A > x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k}$$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k-1} \geq 1 + B > y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k}$$

Case 1: if $x_{n_0}, x_{n_0+2}, ..., x_{n_0+k-1} < 1 + A \leq x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k}$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k-1} < 1 + B \leq y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k}$$

then

$$x_{n_0+k+1} = A + \frac{y_{n_0}}{y_{n_0+k}} < A + 1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0}}{x_{n_0+k}} < B + 1$$

so (x_{n_0+k}, y_{n_0+k}) is the k^{th} semi-cycle of length one. By induction, assume

[1] x_{n_0} is the last term in the previous semi-cycle which we have no information about its length, and x_{n_0+k} is the first term in the next semi-cycle which we have no information about its length.

there are $k-1+m$ semi-cycle of length one. If m is odd, then

$$x_{n_0}, x_{n_0+2}, ..., x_{n_0+k+m} < 1+A \leq x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k+m+1}$$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k+m} < 1+B \leq y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k+m+1}$$

then

$$x_{n_0+k+m+2} = A + \frac{y_{n_0+m+1}}{y_{n_0+k+m+1}} < A+1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0+m+1}}{x_{n_0+k+m+1}} < B+1$$

so every semi-cycle is of length one. If m is even, then

$$x_{n_0}, x_{n_0+2}, ..., x_{n_0+k+m+1} < 1+A \leq x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k+m}$$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k+m+1} < 1+B \leq y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k+m}$$

then

$$x_{n_0+k+m+2} = A + \frac{y_{n_0+m+1}}{y_{n_0+k+m+1}} > A+1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0+m+1}}{x_{n_0+k+m+1}} > B+1$$

so every semi-cycle is of length one.

Case 2: if $x_{n_0}, x_{n_0+2}, ..., x_{n_0+k-1} \geq 1+A > x_{n_0+1}, x_{n_0+3}, ..., x_{n_0+k}$

and

$$y_{n_0}, y_{n_0+2}, ..., y_{n_0+k-1} \geq 1+B > y_{n_0+1}, y_{n_0+3}, ..., y_{n_0+k}$$

then

$$x_{n_0+k+1} = A + \frac{y_{n_0}}{y_{n_0+k}} > A + 1$$

and

$$y_{n_0+k+1} = B + \frac{x_{n_0}}{x_{n_0+k}} > B + 1$$

so (x_{n_0+k}, y_{n_0+k}) is the k^{th} semi-cycle of length one. By induction, every semi-cycle after this point is of length one. The proof is complete. □

Theorem 2.1.3. *System (2.1) has no nontrivial k-periodic solutions (not necessarily prime period k).*

Proof. Assume system (2.1) has a k-periodic solution. Then, $(x_{n-k}, y_{n-k}) = (x_n, y_n)$ for all $n \geq 0$, and so

$$x_{n+1} = A + \frac{y_{n-k}}{y_n} = A + 1, \text{ and } y_{n+1} = B + \frac{x_{n-k}}{x_n} = B + 1, \ n \geq 0.$$

Thus, the solution $(x_n, y_n) = (A + 1, B + 1)$ is the equilibrium solution of (2.1). The proof is complete. □

Theorem 2.1.4. *All non-oscillatory solutions of system (2.1) tends to the equilibrium $(\bar{x}, \bar{y}) = (A + 1, B + 1)$ as $n \to \infty$.*

Proof. Assume system (2.1) has a non-oscillatory solution, say $\{x_n, y_n\}_{n=-k}^{\infty}$. Then by Theorem 2.1.1 the solution consists of a single semi-cycle, either this semi-cycle is positive or negative. Assume that the solution is of a positive semi-cycle. Then for all $n \geq -k$, $(x_n, y_n) \geq (A + 1, B + 1)$, so

$$x_{n+1} = A + \frac{y_{n-k}}{y_n} \geq A + 1 \quad \text{implies} \quad y_{n-k} \geq y_n$$

and

$$y_{n+1} = B + \frac{x_{n-k}}{x_n} \geq B + 1 \quad \text{implies} \quad x_{n-k} \geq x_n$$

so

$$x_{n-k} \geq x_n \geq x_{n+k} \geq ... \geq A+1 \text{ and } y_{n-k} \geq y_n \geq y_{n+k} \geq ... \geq B+1, \quad n \geq 0$$

which means that $\{x_n\}, \{y_n\}$ have k subsequences

$$\{x_{nk}\}, \{x_{nk+1}\}, ..., \{x_{nk+(k-1)}\} \text{ and } \{y_{nk}\}, \{y_{nk+1}\}, ..., \{y_{nk+(k-1)}\}$$

each subsequence is decreasing and bounded from below, so each one of them is convergent, so for all $i = 0, 1, ..., k-1$ there exist α_i, β_i such that

$$\lim_{n\to\infty} x_{nk+i} = \alpha_i \text{ and } \lim_{n\to\infty} y_{nk+i} = \beta_i$$

Thus,

$$(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{k-1}, \beta_{k-1})$$

is a k-periodic solution of system (2.1), which contradicts Theorem 2.1.3 unless the solution is the trivial solution. Hence, the solution converges to the equilibrium. □

2.2 The Case $0 < A < 1$ and $0 < B < 1$

In this section, we study the asymptotic behavior of the positive solutions of system (2.1) when $0 < A < 1$ and $0 < B < 1$.

Theorem 2.2.1. *Suppose that $0 < A < 1$ and $0 < B < 1$. Let $C = \max\{A, B\}$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ be an arbitrary positive solution of (2.1). Then* *the following statements are true:*

1. *If k is odd and $0 < x_{2m-1} < 1$, $0 < y_{2m-1} < 1$, $x_{2m} > \frac{1}{1-C}$, $y_{2m} > \frac{1}{1-C}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then*

$$\lim_{n\to\infty} x_{2n} = \infty, \lim_{n\to\infty} y_{2n} = \infty, \lim_{n\to\infty} x_{2n+1} = A, \lim_{n\to\infty} y_{2n+1} = B$$

2. *If k is odd and $0 < x_{2m} < 1$, $0 < y_{2m} < 1$, $x_{2m-1} > \frac{1}{1-C}$, $x_{2m-1} > \frac{1}{1-C}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then*

$$\lim_{n\to\infty} x_{2n+1} = \infty, \lim_{n\to\infty} y_{2n+1} = \infty, \lim_{n\to\infty} x_{2n} = A, \lim_{n\to\infty} y_{2n} = B$$

Proof. 1. Since $C \geq A$ and B, it is clear that $1 - C \leq 1 - A$ and $1 - B$. Then

$$\begin{aligned} 0 < x_1 =& A + \frac{y_{-k}}{y_0} < A + \frac{1}{y_0} < A + 1 - C \leq A + 1 - A = 1 \\ 0 < y_1 =& B + \frac{x_{-k}}{x_0} < B + \frac{1}{x_0} < B + 1 - C \leq B + 1 - B = 1 \\ x_2 =& A + \frac{y_{-k+1}}{y_1} > A + y_{-k+1} > y_{-k+1} > \frac{1}{1-C} \\ y_2 =& B + \frac{x_{-k+1}}{x_1} > B + x_{-k+1} > x_{-k+1} > \frac{1}{1-C} \end{aligned}$$

By induction, we have

$$0 < x_{2n-1}, y_{2n-1} < 1, x_{2n}, y_{2n} > \frac{1}{1-C} \text{ for } n = 1, 2, \ldots$$

so for $l > \frac{k+3}{2}$

$$\begin{aligned} x_{2l} &= A + \frac{y_{2l-(k+1)}}{y_{2l-1}} > A + y_{2l-(k+1)} = A + B + \frac{x_{2l-(2k+2)}}{x_{2l-k-2}} \\ &> A + B + x_{2l-(2k+2)} \\ x_{4l} &= A + \frac{y_{4l-(k+1)}}{y_{4l-1}} > A + y_{4l-(k+1)} = A + B + \frac{x_{4l-(2k+2)}}{x_{4l-k-2}} \\ &> A + B + x_{4l-(2k+2)} = 2A + B + \frac{y_{4l-(3k+3)}}{y_{4l-2k-3}} \\ &> 2A + B + y_{4l-(3k+3)} = 2A + 2B + \frac{y_{4l-(4k+4)}}{y_{4l-3k-4}} \\ &> 2A + 2B + x_{4l-(4k+4)} \end{aligned}$$

similarly

$$x_{6l} > 3A + 3B + x_{6l-(6k+6)}$$

so for all $r = 1, 2, \ldots$

$$x_{2rl} > r(A + B) + x_{2rl-2r(k+1)}$$

if $n = rl$, then as $r \to \infty, n \to \infty$, $\lim_{n\to\infty} x_{2n} = \infty$. Similarly, we get $\lim_{n\to\infty} y_{2n} = \infty$. Considering (2.1) and taking the limits on both sides of each equation in the system

$$x_{2n+1} = A + \frac{y_{2n-k}}{y_{2n}}, y_{2n+1} = B + \frac{x_{2n-k}}{x_{2n}}$$

we obtain $\lim_{n\to\infty} x_{2n+1} = A$ and $\lim_{n\to\infty} y_{2n+1} = B$. Hence, we complete the proof of 1.

2. If k is odd, $0 < x_{2m} < 1$, $0 < y_{2m} < 1$, $x_{2m-1} > \frac{1}{1-C}$, $x_{2m-1} > \frac{1}{1-C}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, and $1 - C \le 1 - A$ and $1 - B$ since $C \ge A$ and

B, then it is clear that

$$\begin{aligned}
x_1 =& A + \frac{y_{-k}}{y_0} > A + y_{-k} > y_{-k} > \frac{1}{1-C} \\
y_1 =& B + \frac{x_{-k}}{x_0} > B + x_{-k} > x_{-k} > \frac{1}{1-C} \\
0 < x_2 =& A + \frac{y_{-k+1}}{y_1} < A + \frac{1}{y_1} < A + 1 - C \leq A + 1 - A = 1 \\
0 < y_2 =& B + \frac{x_{-k+1}}{x_1} < B + \frac{1}{x_1} < B + 1 - C \leq B + 1 - B = 1
\end{aligned}$$

Using induction implies that

$$0 < x_{2n}, y_{2n} < 1, x_{2n-1}, y_{2n-1} > \frac{1}{1-C} \quad \text{for} \quad n = 1, 2, \ldots$$

so for $l > \frac{k+1}{2}$

$$\begin{aligned}
x_{2l+1} =& A + \frac{y_{2l-k}}{y_{2l}} > A + y_{2l-k} = A + B + \frac{x_{2l-2k-1}}{x_{2l-k-1}} > A + B + x_{2l-2k-1} \\
x_{4l+1} =& A + \frac{y_{4l-k}}{y_{4l}} > A + y_{4l-k} = A + B + \frac{x_{4l-2k-1}}{x_{4l-k-1}} > A + B + x_{4l-2k-1} \\
=& 2A + B + \frac{y_{4l-3k-2)}}{y_{4l-2k-2}} > 2A + B + y_{4l-3k-2} \\
=& 2A + 2B + \frac{x_{4l-4k-3}}{x_{4l-3k-3}} > 2A + 2B + x_{4l-4k-3}
\end{aligned}$$

similarly

$$x_{6l+1} > 3A + 3B + x_{6l-6k-5}$$

so for all $r = 1, 2, \ldots$

$$x_{2rl+1} > r(A+B) + x_{2rl-2rk-(2r-1)}$$

if $n = rl$, then as $r \to \infty, n \to \infty$, $\lim_{n\to\infty} x_{2n+1} = \infty$. Similarly, we get $\lim_{n\to\infty} y_{2n+1} = \infty$. Considering (2.1) and taking the limits on both sides of each equation in the system

$$x_{2n+2} = A + \frac{y_{2n-k+1}}{y_{2n+1}}, \quad y_{2n+2} = B + \frac{x_{2n-k+1}}{x_{2n+1}}$$

we obtain $\lim_{n\to\infty} x_{2n} = A$ and $\lim_{n\to\infty} y_{2n} = B$.

The proof is complete. □

Remark 2.2.1. Note that when $A = B = 1$, then system (2.1) is of the form

$$x_{n+1} = 1 + \frac{y_{n-k}}{y_n}, y_{n+1} = 1 + \frac{x_{n-k}}{x_n}$$

which was studied by Zhang et al. [22].

2.3 The Case $A > 1$ and $B > 1$

In this section, we study the boundedness and persistence of the positive solutions of system (2.1) when $A > 1$ and $B > 1$, we also prove that if $A > 1$ and $B > 1$ then the unique positive equilibrium of (2.1) is globally asymptotically stable.

Lemma 2.3.1. *Given v_j, where $j = -k, -k+1, \ldots, k+1$. Then the solution of the higher order linear difference equation*

$$v_{n+2k+2} = av_n + b, \quad n \geq -k, a \neq 1$$

is of the form

$$v_{i+l(2k+2)} = \left(v_{i-(2k+2)} + \frac{b}{a-1}\right) a^{l+1} + \frac{b}{1-a}$$

for $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$.

Proof.

$$\text{when } n = -k, \quad v_{k+2} = av_{-k} + b$$
$$\text{when } n = -k+1, \quad v_{k+3} = av_{-k+1} + b$$
$$\vdots$$
$$\text{when } n = k+1, \quad v_{3k+3} = av_{k+1} + b$$

$$\text{Moreover, when } n = k+2, \quad v_{3k+4} = av_{k+2} + b = a^2 v_{-k} + ab + b$$
$$\text{when } n = k+3, \quad v_{3k+5} = av_{k+3} + b = a^2 v_{-k+1} + ab + b$$
$$\vdots$$
$$\text{when } n = 3k+3, \quad v_{5k+5} = av_{3k+3} + b = a^2 v_{k+1} + ab + b$$
$$\vdots$$

hence, for $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$

$$\begin{aligned} v_{i+l(2k+2)} &= a^{l+1} v_{i-(2k+2)} + b(a^l + a^{l-1} + \cdots + 1) \\ &= \left(v_{i-(2k+2)} + \frac{b}{a-1} \right) a^{l+1} + \frac{b}{1-a} \end{aligned}$$

which completes the proof. □

Theorem 2.3.2. *Suppose that $A > 1$ and $B > 1$. Then every positive solution of system (2.1) is bounded and persists. In particular, for $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$, every positive solution of system (2.1) satisfies*

$$A < x_{i+l(2k+2)} < \left(x_{i-(2k+2)} + \frac{(A+1)AB}{1-AB} \right) \left(\frac{1}{AB} \right)^{l+1} + \frac{(A+1)AB}{AB-1}$$

$$B < y_{i+l(2k+2)} < \left(y_{i-(2k+2)} + \frac{(B+1)AB}{1-AB} \right) \left(\frac{1}{AB} \right)^{l+1} + \frac{(B+1)AB}{AB-1}$$

Proof. Assume $A > 1,\ B > 1$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of system (2.1). Since $x_n > 0$ and $y_n > 0$ for all $n \geq -k$, (2.1) implies that

$$x_n > A > 1, y_n > B > 1 \text{ for all } n \geq 1 \tag{2.2}$$

Now, using (2.1) and (2.2) we get that for all $n \geq 2$

$$\begin{aligned} x_n &= A + \frac{y_{n-k-1}}{y_{n-1}} < A + \frac{1}{B} y_{n-k-1} \\ y_n &= B + \frac{x_{n-k-1}}{x_{n-1}} < B + \frac{1}{A} x_{n-k-1} \end{aligned} \tag{2.3}$$

Let $\{v_n, w_n\}$ be the solution of the system

$$v_{n+1} = A + \frac{1}{B} w_{n-k}, \quad w_{n+1} = B + \frac{1}{A} v_{n-k} \text{ for all } n \geq k+1 \tag{2.4}$$

such that

$$v_i = x_i, \quad w_i = y_i, \quad i = -k, -k+1, \ldots, 0, 1, \ldots, k+1 \tag{2.5}$$

now, we use induction to prove that

$$x_n < v_n, \quad y_n < w_n, \quad n \geq k+2 \tag{2.6}$$

Suppose that (2.6) is true for $n = m \geq k+2$. Then, from (2.3), we get

$$\begin{aligned} x_{m+1} &< A + \frac{1}{B} y_{m-k} < A + \frac{1}{B} w_{m-k} = v_{m+1} \\ y_{m+1} &< B + \frac{1}{A} x_{m-k} < B + \frac{1}{A} v_{m-k} = w_{m+1} \end{aligned}$$

Therefore, (2.6) is true. From (2.4) and (2.5), we have

$$v_{n+2k+2} = \frac{1}{AB} v_n + A + 1, \quad w_{n+2k+2} = \frac{1}{AB} w_n + B + 1, \quad n \geq -k \tag{2.7}$$

for simplicity, let $a = \frac{1}{AB}$, $b = A + 1$ and $c = B + 1$. Then (2.7) becomes

$$v_{n+2k+2} = av_n + b, \quad w_{n+2k+2} = aw_n + c, \quad n \geq -k$$

Now, using Lemma 2.3.1, for all $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$

$$\begin{aligned} v_{i+l(2k+2)} &= a^{l+1} x_{i-(2k+2)} + b(a^l + a^{l-1} + \cdots + 1) \\ &= \left(x_{i-(2k+2)} + \frac{b}{a-1} \right) a^{l+1} + \frac{b}{1-a} \end{aligned}$$

since $A > 1$, $B > 1$ and $a = \frac{1}{AB}$, $b = A+1$. Then for $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$

$$v_{i+l(2k+2)} = \left(x_{i-(2k+2)} + \frac{(A+1)AB}{1-AB} \right) \left(\frac{1}{AB} \right)^{l+1} + \frac{(A+1)AB}{AB-1} \tag{2.8}$$

Then, from (2.2), (2.6), and (2.8), for all $i = k+2, k+3, \ldots, 3k+3$ and $l \geq 0$

$$A < x_{i+l(2k+2)} < \left(x_{i-(2k+2)} + \frac{(A+1)AB}{1-AB} \right) \left(\frac{1}{AB} \right)^{l+1} + \frac{(A+1)AB}{AB-1}$$

Similarly, we get

$$B < y_{i+l(2k+2)} < \left(y_{i-(2k+2)} + \frac{(B+1)AB}{1-AB} \right) \left(\frac{1}{AB} \right)^{l+1} + \frac{(B+1)AB}{AB-1}$$

The proof is complete. □

Theorem 2.3.3. *If $A > 1$ and $B > 1$, then every positive solution of system (2.1) converges to the equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ as $n \to \infty$.*

Proof. Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be an arbitrary positive solution of (2.1), and let

$$\begin{aligned} u_1 &= \limsup_{n\to\infty} x_n, \quad l_1 = \liminf_{n\to\infty} x_n \\ u_2 &= \limsup_{n\to\infty} y_n, \quad l_2 = \liminf_{n\to\infty} y_n \end{aligned}$$

Using previous theorem, we have $0 < A \leq l_1 \leq u_1 < +\infty$ and $0 < B \leq l_2 \leq u_2 < +\infty$. Now, system (2.1) implies that

$$u_1 \leq A + \frac{u_2}{l_2}, \quad u_2 \leq B + \frac{u_1}{l_1}, \quad l_1 \geq A + \frac{l_2}{u_2}, \quad l_2 \geq B + \frac{l_1}{u_1} \tag{2.9}$$

then

$$Bu_1 + l_1 \leq u_1 l_2 \leq Al_2 + u_2 \tag{2.10}$$

$$Au_2 + l_2 \leq u_2 l_1 \leq Bl_1 + u_1 \tag{2.11}$$

from (2.10) we get

$$Bu_1 + l_1 \leq Al_2 + u_2 \tag{2.12}$$

and (2.11) implies

$$-Bl_1 - u_1 \leq -Au_2 - l_2 \tag{2.13}$$

from (2.12) and (2.13) we get

$$Bu_1 + l_1 - Bl_1 - u_1 \leq Al_2 + u_2 - Au_2 - l_2$$

and

$$(B-1)(u_1 - l_1) + (A-1)(u_2 - l_2) \leq 0$$

but $A, B > 1$ so $A - 1, B - 1 > 0$, also $u_1 - l_1, u_2 - l_2 \geq 0$. Hence

$$u_1 - l_1 = 0 \ \text{ and } \ u_2 - l_2 = 0$$

so $u_1 = l_1$ and $u_2 = l_2$. Now use (2.9) to get

$$B + 1 \leq l_2 = u_2 \leq B + 1 \quad \text{and} \quad A + 1 \leq l_1 = u_1 \leq A + 1$$

hence

$$l_1 = u_1 = A + 1 \quad \text{and} \quad l_2 = u_2 = B + 1$$

so

$$\lim_{n\to\infty} x_n = l_1 = u_1 = A + 1 \quad \text{and} \quad \lim_{n\to\infty} y_n = l_2 = u_2 = B + 1$$

which complete the proof. □

Lemma 2.3.4. *If $A > 1$ and $0 < \varepsilon < \frac{A-1}{(A+1)(k+1)}$ where $k \in \mathbb{Z}^+$, then $\frac{2}{(1-(k+1)\varepsilon)(A+1)} < 1$.*

Proof.

$$0 < \varepsilon < \frac{1}{(k+1)}\frac{A-1}{A+1} \quad \text{implies} \quad 0 < (k+1)\varepsilon < \frac{A-1}{A+1}$$

so

$$1 - (k+1)\varepsilon > 1 - \frac{A-1}{A+1} = \frac{2}{A+1}$$

that is,

$$\frac{1}{1-(k+1)\varepsilon} < \frac{A+1}{2} \quad \text{implies} \quad \frac{2}{1-(k+1)\varepsilon} < A + 1$$

and so

$$\frac{2}{(1-(k+1)\varepsilon)(A+1)} < 1$$

The proof is complete. □

Now, we'll prove that the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.1) is locally asymptotically stable using the previous lemma.

Theorem 2.3.5. *If $A > 1$ and $B > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.1) is locally asymptotically stable.*

Proof. System (2.1) can be formulated as a system of first order recurrence equations as follows:

$$\begin{aligned} w_n^{(1)} =& x_n, w_n^{(2)} = x_{n-1}, \ldots, w_n^{(k+1)} = x_{n-k} \\ v_n^{(1)} =& y_n, v_n^{(2)} = y_{n-1}, \ldots, v_n^{(k+1)} = y_{n-k} \end{aligned} \tag{2.14}$$

Let $Z_n = (w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(k+1)}, v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(k+1)})^T$, Then the linearized equation of system (2.1) associated with (2.14) about the equilibrium point $(\bar{x}, \bar{y}) = (A+1, B+1)$ is

$$Z_{n+1} = JZ_n$$

where

$$Z_{n+1} = \begin{pmatrix} w_{n+1}^{(1)} \\ w_{n+1}^{(2)} \\ \vdots \\ w_{n+1}^{(k+1)} \\ v_{n+1}^{(1)} \\ v_{n+1}^{(2)} \\ \vdots \\ v_{n+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} A + \frac{v_n^{(k+1)}}{v_n^{(1)}} \\ w_n^{(1)} \\ \vdots \\ w_n^{(k)} \\ B + \frac{w_n^{(k+1)}}{w_n^{(1)}} \\ v_n^{(1)} \\ \vdots \\ v_n^{(k)} \end{pmatrix}$$

and J is the Jacobian matrix.

$$\begin{aligned} & J_{(2k+2)\times(2k+2)} \\ = & \left(D_{w_n^{(1)}} Z_{n+1} \quad \ldots \quad D_{w_n^{(k+1)}} Z_{n+1} \quad D_{v_n^{(1)}} Z_{n+1} \quad \ldots \quad D_{v_n^{(k+1)}} Z_{n+1} \right) \end{aligned}$$

so the Jacobian matrix will be of the following form

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{-1}{B+1} & 0 & \dots & 0 & \frac{1}{B+1} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-1}{A+1} & 0 & \dots & 0 & \frac{1}{A+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Let $\lambda_1, \lambda_2, ..., \lambda_{2k+2}$ be the eigenvalues of J. Define $D = diag(d_1, d_2, ..., d_{2k+2})$ be a diagonal matrix such that

$$d_1 = d_{k+2} = 1, \;\; d_m = d_{k+1+m} = 1 - m\varepsilon, \;\; m = 2, 3, ..., k+1$$

choose $\varepsilon > 0$ such that $0 < \varepsilon < \min\{\frac{A-1}{(A+1)(k+1)}, \frac{B-1}{(B+1)(k+1)}\}$. Now,

$$D_{(2k+2)\times(2k+2)} = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-2\varepsilon & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1-(k+1)\varepsilon & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1-2\varepsilon & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1-(k+1)\varepsilon \end{pmatrix}$$

so for all $m = 2, 3, \dots, k+1$,

$$1 - m\varepsilon \geq 1 - (k+1)\varepsilon > 1 - \frac{(k+1)(A-1)}{(k+1)(A+1)} = \frac{A+1-A+1}{A+1} = \frac{2}{A+1} > 0$$

so for all m, $1 - m\varepsilon > 0$, hence D is invertible. Now,

$$DJD^{-1}_{(2k+2)\times(2k+2)} =$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{-1}{B+1}\frac{d_1}{d_{k+2}} & 0 & \dots & 0 & \frac{1}{B+1}\frac{d_1}{d_{2k+2}} \\ \frac{d_2}{d_1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{d_{k+1}}{d_k} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-1}{A+1}\frac{d_{k+2}}{d_1} & 0 & \dots & 0 & \frac{1}{A+1}\frac{d_{k+2}}{d_{k+1}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{d_{2k+2}}{d_{2k+1}} & 0 \end{pmatrix}$$

Now, we want to show that the sum of the absolute value of entries of each row is less than one, in order to find the infinite norm of DJD^{-1}.

Since $\varepsilon > 0$ so $1 - m\varepsilon > 1 - (m+1)\varepsilon$, that is, $d_m > d_{m+1}$, for all m. So

$$\frac{d_2}{d_1} < 1, \quad \frac{d_3}{d_2} < 1, \ldots, \quad \frac{d_{2k+2}}{d_{2k+1}} < 1$$

For
$$\begin{aligned}\frac{1}{B+1}\frac{d_1}{d_{k+2}} + \frac{1}{B+1}\frac{d_1}{d_{2k+2}} &= \frac{1}{B+1} + \frac{1}{(1-(k+1)\varepsilon)(B+1)} \\ &< \frac{1}{1-(k+1)\varepsilon}\frac{1}{(B+1)} + \frac{1}{1-(k+1)\varepsilon}\frac{1}{(B+1)} \\ &= \frac{2}{(1-(k+1)\varepsilon)(B+1)} \qquad \text{use Lemma 2.3.4} \\ &< 1\end{aligned}$$

For
$$\begin{aligned}\frac{1}{A+1}\frac{d_{k+2}}{d_1} + \frac{1}{A+1}\frac{d_{k+2}}{d_{k+1}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\ &< \frac{1}{1-(k+1)\varepsilon}\frac{1}{(A+1)} + \frac{1}{1-(k+1)\varepsilon}\frac{1}{(A+1)} \\ &= \frac{2}{(1-(k+1)\varepsilon)(A+1)} \qquad \text{use Lemma 2.3.4} \\ &< 1\end{aligned}$$

Since J has the same eigenvalue as DJD^{-1}. Then,

$$\rho(J) = \max\{|\lambda_i|\} \leq \|DJD^{-1}\|_\infty$$

but

$$\|DJD^{-1}\|_\infty = \max\left\{\begin{array}{c}\frac{1}{B+1} + \frac{1}{(1-(k+1)\varepsilon)(B+1)}, \frac{d_2}{d_1}, \frac{d_3}{d_2}, \ldots, \frac{d_{k+1}}{d_k}, \\ \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)}\end{array}\right\} < 1$$

So the modulus of every eigenvalue of J is less than one. Hence, the unique equilibrium point $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.1) is locally asymptotically stable. □

Theorem 2.3.6. *If $A > 1$ and $B > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A + 1, B + 1)$ of system (2.1) is globally asymptotically stable.*

Proof. Using Theorem 2.3.5, we conclude that the equilibrium $(\bar{x}, \bar{y}) = (A + 1, B + 1)$ of system (2.1) is asymptotically stable, but Theorem 2.3.3 implies that this equilibrium is a global attractor. Thus, the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.1) is globally asymptotically stable. □

2.4 A Special Case $k = 2$

In this section, we give useful theorems to understand the behavior of solutions when $k = 2$ in system (2.1), so it turns into the following:

$$x_{n+1} = A + \frac{y_{n-2}}{y_n}, y_{n+1} = B + \frac{x_{n-2}}{x_n}, n = 0, 1, \ldots \tag{2.15}$$

Theorem 2.4.1. *With the exception of possibly the first semi-cycle, all semi-cycles of system (2.15) have one or two terms.*

Proof. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of system (2.15). Assume there exists $n_0 > 0$ such that either

$$x_{n_0-1} \geq A + 1 > x_{n_0} \text{ and } y_{n_0-1} \geq B + 1 > y_{n_0}$$

or

$$x_{n_0-1} < A + 1 \leq x_{n_0} \text{ and } y_{n_0-1} < B + 1 \leq y_{n_0}$$

If $x_{n_0-1} \geq A+1 > x_{n_0}$ and $y_{n_0-1} \geq B+1 > y_{n_0}$, then either $x_{n_0+1} \geq A+1$ and $y_{n_0+1} \geq B+1$ so the semi cycle is of length one, or $x_{n_0+1} < A+1$ and $y_{n_0+1} < B+1$ then $x_{n_0+2} = A + \frac{y_{n_0-1}}{y_{n_0+1}} > A+1$ and $y_{n_0+2} = B + \frac{x_{n_0-1}}{x_{n_0+1}} > B+1$ so the semi cycle is of length two.

If $x_{n_0-1} < A+1 \leq x_{n_0}$ and $y_{n_0-1} < B+1 \leq y_{n_0}$, then either $x_{n_0+1} < A+1$ and $y_{n_0+1} < B+1$ so the semi cycle is of length one, or $x_{n_0+1} \geq A+1$ and $y_{n_0+1} \geq B+1$ then $x_{n_0+2} = A + \frac{y_{n_0-1}}{y_{n_0+1}} < A+1$ and $y_{n_0+2} = B + \frac{x_{n_0-1}}{x_{n_0+1}} < B+1$ so the semi cycle is of length two. This completes the proof. □

Theorem 2.4.2. *Consider system (2.15). Every positive semi-cycle of length two is followed by a negative semi-cycle of length one and every negative semi-cycle of length two is followed by a positive semi-cycle of length one.*

Proof. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of system (2.15). Assume there exists $n_0 > 0$ such that either

$$x_{n_0-1} < A+1 \leq x_{n_0}, x_{n_0+1} \text{ and } x_{n_0+2} < A+1$$

and

$$y_{n_0-1} < B+1 \leq y_{n_0}, y_{n_0+1} \text{ and } y_{n_0+2} < B+1$$

or

$$x_{n_0-1} \geq A+1 > x_{n_0}, x_{n_0+1} \text{ and } x_{n_0+2} \geq A+1$$

and

$$y_{n_0-1} \geq B+1 > y_{n_0}, y_{n_0+1} \text{ and } y_{n_0+2} \geq B+1$$

If

$$x_{n_0-1} < A+1 \leq x_{n_0}, x_{n_0+1} \text{ and } x_{n_0+2} < A+1$$

and

$$y_{n_0-1} < B + 1 \leq y_{n_0}, y_{n_0+1} \ \text{ and } \ y_{n_0+2} < B + 1$$

then

$$\frac{x_{n_0}}{x_{n_0+2}} > 1 \ \text{ and } \ \frac{y_{n_0}}{y_{n_0+2}} > 1$$

So,

$$x_{n_0+3} = A + \frac{y_{n_0}}{y_{n_0+2}} > A + 1$$

and

$$y_{n_0+3} = B + \frac{x_{n_0}}{x_{n_0+2}} > B + 1$$

Hence, every positive semi-cycle of length two is followed by a negative semi-cycle of length one.

If

$$x_{n_0-1} \geq A + 1 > x_{n_0}, x_{n_0+1} \ \text{ and } \ x_{n_0+2} \geq A + 1$$

and

$$y_{n_0-1} \geq B + 1 > y_{n_0}, y_{n_0+1} \ \text{ and } \ y_{n_0+2} \geq B + 1$$

then

$$\frac{x_{n_0}}{x_{n_0+2}} < 1 \ \text{ and } \ \frac{y_{n_0}}{y_{n_0+2}} < 1$$

So,

$$x_{n_0+3} = A + \frac{y_{n_0}}{y_{n_0+2}} < A + 1$$

and

$$y_{n_0+3} = B + \frac{x_{n_0}}{x_{n_0+2}} < B + 1$$

Hence, every negative semi-cycle of length two is followed by a positive semi-cycle of length one. □

Theorem 2.4.3. *Suppose that there exists $n_0 > 1$ such that (x_{n_0}, y_{n_0}) is the single term in a positive semi-cycle of length one. Then $x_{n_0+2} \in (A + \frac{B}{B+1}, A + \frac{B+1}{B})$, and $y_{n_0+2} \in (B + \frac{A}{A+1}, B + \frac{A+1}{A})$.*

Proof. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of system (2.15). Assume that there exists $n_0 > 1$ such that (x_{n_0}, y_{n_0}) is the single term in a positive semi-cycle of length one. Then

$$A < x_{n_0+1}, x_{n_0-1} < A + 1 \leq x_{n_0} \text{ and } B < y_{n_0+1}, y_{n_0-1} < B + 1 \leq y_{n_0}$$

Hence,

$$A + \frac{B}{B+1} < x_{n_0+2} = A + \frac{y_{n_0-1}}{y_{n_0+1}} < A + \frac{B+1}{B}$$

and

$$B + \frac{A}{A+1} < y_{n_0+2} = B + \frac{x_{n_0-1}}{x_{n_0+1}} < B + \frac{A+1}{A}$$

The proof is complete. □

Theorem 2.4.4. *Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of system (2.15). Then $\frac{x_{n+3}}{y_n} < \frac{A+1}{B}$ and $\frac{y_{n+3}}{x_n} < \frac{B+1}{A}$ for $n \geq 1$.*

Proof. Let $\{x_n, y_n\}_{n=-2}^{\infty}$ be a solution of system (2.15). Then

$$x_{n+3} = A + \frac{y_n}{y_{n+2}} \text{ implies } \frac{x_{n+3}}{y_n} = \frac{A}{y_n} + \frac{1}{y_{n+2}} < \frac{A}{B} + \frac{1}{B} = \frac{A+1}{B}$$

and

$$y_{n+3} = B + \frac{x_n}{x_{n+2}} \text{ implies } \frac{y_{n+3}}{x_n} = \frac{B}{x_n} + \frac{1}{x_{n+2}} < \frac{B}{A} + \frac{1}{A} = \frac{B+1}{A}$$

The result then follows. □

Lemma 2.4.5. *System (2.1) has no nontrivial eventually k-periodic solutions (not necessarily prime period k).*

Proof. Assume system (2.1) has a k-periodic solution with delay by l terms. Then, $(x_{n+l}, y_{n+l}) = (x_{n+l+k}, y_{n+l+k})$ for all $n \geq 0$, and so

$$x_{n+l+k+1} = A + \frac{y_{n+l}}{y_{n+l+k}} = A + 1, \text{ and } y_{n+l+k+1} = B + \frac{x_{n+l}}{x_{n+l+k}} = B + 1$$

Then the eventually k-periodic solution of system (2.1) with delay l is the equilibrium of (2.1) with delay l or $l+1$, that is, if $(x_l, y_l) = (x_{l+k}, y_{l+k}) \neq (A+1, B+1)$, then the solution is the equilibrium of (2.1) with delay $l+1$, and if $(x_l, y_l) = (x_{l+k}, y_{l+k}) = (A+1, B+1)$, then the solution is the equilibrium of (2.1) with delay l. This completes the proof. □

Theorem 2.4.6. *Every solution of system (2.15) that oscillates has infinitely many semi-cycles of length two.*

Proof. On the contrary, assume that system (2.15) has a solution say $\{x_n, y_n\}_{n=-2}^{\infty}$ that oscillates and has finitely many semi-cycles of length two. Then every semi-cycle after that point is of length one.

Assume that the last term in the last semi-cycle of length two is (x_{n_0}, y_{n_0}). Then,

Case 1: if $n_0 = 2k_0$ for some $k_0 \in \mathbb{Z}^+$, and the semi-cycle containing (x_{n_0}, y_{n_0}) is a positive semi-cycle, then

$$x_{2n} \geq A + 1 > x_{2n+1} \text{ and } y_{2n} \geq B + 1 > y_{2n+1} \text{ for all } n \geq k_0$$

Now,

$$x_{2n+2} = A + \frac{y_{2n-1}}{y_{2n+1}} \geq A + 1 \text{ for all } n > k_0 \text{ implies } y_{2n-1} \geq y_{2n+1}$$

but for all $n, y_n > B$, so $B < y_{2n+1} \leq y_{2n-1} < B + 1$ for all $n > k_0$. Also

$$y_{2n+2} = B + \frac{x_{2n-1}}{x_{2n+1}} \geq B + 1 \text{ for all } n > k_0 \text{ implies } x_{2n-1} \geq x_{2n+1}$$

but for all $n, x_n > A$, so $A < x_{2n+1} \leq x_{2n-1} < 1 + A$ for all $n > k_0$. Moreover, for all $n > k_0$, $x_{2n+2} < A + \frac{B+1}{B}$ and $y_{2n+2} < B + \frac{A+1}{A}$. Also

$$x_{2n+3} = A + \frac{y_{2n}}{y_{2n+2}} < A + 1 \text{ for all } n > k_0 \text{ implies } y_{2n} < y_{2n+2}$$

and

$$y_{2n+3} = B + \frac{x_{2n}}{x_{2n+2}} < B + 1 \text{ for all } n > k_0 \text{ implies } x_{2n} < x_{2n+2}$$

moreover, for all $n > k_0$, $A + 1 \leq x_{2n} < x_{2n+2} < A + \frac{B+1}{B}$ and $B + 1 \leq y_{2n} < y_{2n+2} < B + \frac{A+1}{A}$.

Hence, the following occurs;

$$A < \cdots \leq x_{2k_0+3} \leq x_{2k_0+1} < A + 1 \leq x_{2k_0} < x_{2k_0+2} \cdots < A + \frac{B+1}{B}$$

$$B < \cdots \leq y_{2k_0+3} \leq y_{2k_0+1} < B + 1 \leq y_{2k_0} < y_{2k_0+2} \cdots < B + \frac{A+1}{A}$$

so there exist finite limits;

$$A + 1 \leq \lim_{n\to\infty} x_{2n} = a < A + \frac{B+1}{B}$$

$$B + 1 \leq \lim_{n\to\infty} y_{2n} = b < B + \frac{A+1}{A}$$

$$A < \lim_{n\to\infty} x_{2n+1} = c < A + 1$$

$$B < \lim_{n\to\infty} y_{2n+1} = d < B + 1$$

So system (2.15) has an eventually two periodic solution of the form

$$\ldots, (c, d), (a, b), (c, d), (a, b), \ldots$$

which contradicts the previous lemma, unless this solution is the equilibrium, but the limits diverges from the equilibrium. Hence, the result follows.

Case 2: if $n_0 = 2k_0 + 1$ for some $k_0 \in \mathbb{Z}^+$, and the semi-cycle containing (x_{n_0}, y_{n_0}) is a positive semi-cycle, then

$$x_{2n-1} \geq 1 + A > x_{2n} \text{ and } y_{2n-1} \geq 1 + B > y_{2n} \text{ for all } n \geq k_0$$

Now,

$$x_{2n+2} = A + \frac{y_{2n-1}}{y_{2n+1}} < A+1 \text{ for all } n > k_0+1 \text{ implies } B+1 \leq y_{2n-1} < y_{2n+1}$$

also

$$y_{2n+2} = B + \frac{x_{2n-1}}{x_{2n+1}} < B+1 \text{ for all } n > k_0+1 \text{ implies } A+1 \leq x_{2n-1} < x_{2n+1}$$

also

$$x_{2n+3} = A + \frac{y_{2n}}{y_{2n+2}} \geq A + 1 \text{ for all } n > k_0 + 1 \text{ implies } y_{2n} \geq y_{2n+2}$$

and

$$y_{2n+3} = B + \frac{x_{2n}}{x_{2n+2}} \geq B + 1 \text{ for all } n > k_0 + 1 \text{ implies } x_{2n} \geq x_{2n+2}$$

but for all n, $y_n > B$ and $x_n > A$, so for all $n > k_0 + 1$, $B + 1 > y_{2n} \geq y_{2n+2} > B$ and $A + 1 > x_{2n} \geq x_{2n+2} > A$, also $x_{2n+3} < A + \frac{B+1}{B}$ and

$y_{2n+3} < B + \frac{A+1}{A}$, which implies $A + 1 \leq x_{2n-1} < x_{2n+1} < A + \frac{B+1}{B}$ and $B + 1 \leq y_{2n-1} < y_{2n+1} < B + \frac{A+1}{A}$ for all $n > k_0 + 1$. Hence, the following occurs;

$$A < \cdots \leq x_{2k_0+4} \leq x_{2k_0+2} < A + 1 \leq x_{2k_0+1} < x_{2k_0+3} \cdots < A + \frac{B+1}{B}$$

$$B < \cdots \leq y_{2k_0+4} \leq y_{2k_0+2} < B + 1 \leq y_{2k_0+1} < y_{2k_0+3} \cdots < B + \frac{A+1}{A}$$

which leads to the same as case 1, that is, there exist finite limits

$$A < \lim_{n\to\infty} x_{2n} = a < A + 1$$

$$B < \lim_{n\to\infty} y_{2n} = b < B + 1$$

$$A + 1 \leq \lim_{n\to\infty} x_{2n+1} = c < A + \frac{B+1}{B}$$

$$B + 1 \leq \lim_{n\to\infty} y_{2n+1} = d < B + \frac{A+1}{A}$$

so system (2.15) has an eventually two periodic solution of the form

$$\ldots, (a, b), (c, d), (a, b), (c, d), \ldots$$

which contradicts the previous lemma.

Case 3: if $n_0 = 2k_0$ for some $k_0 \in \mathbb{Z}^+$, and the semi-cycle containing (x_{n_0}, y_{n_0}) is a negative semi-cycle, then

$$x_{2n} < A + 1 \leq x_{2n+1} \text{ and } y_{2n} < B + 1 \leq y_{2n+1} \text{ for all } n \geq k_0$$

which leads to case 2.

Case 4: if $n_0 = 2k_0 + 1$ for some $k_0 \in \mathbb{Z}^+$, and the semi-cycle containing

(x_{n_0}, y_{n_0}) is a negative semi-cycle, then

$$x_{2n-1} < A+1 \le x_{2n} \text{ and } y_{2n-1} < B+1 \le y_{2n} \text{ for all } n \ge k_0+1$$

which leads to case 1.

Hence, every oscillatory solution of system (2.15) has infinitely many semi-cycles of length two. □

2.5 Numerical Examples

In this section, we give several numerical examples that represent different cases of dynamical behavior of solutions of (2.1) using MATLAB to support the results we had in the previous sections.

Example 2.5.1. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{y_{n-5}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-5}}{x_n}, \quad n = 0, 1, \dots \tag{2.16}$$

with $A = 0.1$, $B = 0.9$, and the initial conditions $x_{-5} = 0.5$, $x_{-4} = 10.1$, $x_{-3} = 0.1$, $x_{-2} = 10.2$, $x_{-1} = 0.2$, $x_0 = 11$, $y_{-5} = 0.2$, $y_{-4} = 11.3$, $y_{-3} = 0.3$, $y_{-2} = 10.3$, $y_{-1} = 0.1$, $y_0 = 12.9$. Then the solution of system (2.16) is unbounded since $0 < A < 1$ and $0 < B < 1$ and the initial conditions satisfy the conditions in Theorem 2.2.1, and the unique positive equilibrium $(\bar{x}, \bar{y}) = (1.1, 1.9)$ is not globally asymptotically stable (see Figure 2.1, Theorem 2.2.1).

Example 2.5.2. Consider system (2.16) with $A = 3$, $B = 1.5$, and the initial conditions $x_{-5} = 2.5$, $x_{-4} = 3.7$, $x_{-3} = 1.5$, $x_{-2} = 0.7$, $x_{-1} = 0.5$, $x_0 =$

0.2, $y_{-5} = 2.2$, $y_{-4} = 3.3$, $y_{-3} = 1.2$, $y_{-2} = 0.3$, $y_{-1} = 0.2$, $y_0 = 0.9$. Since $A > 1$ and $B > 1$, the solution of system (2.16) is bounded and persists (see Theorem 2.3.2), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (4, 2.5)$ is globally asymptotically stable (see Figure 2.2, Theorem 2.3.6).

Example 2.5.3. Consider the following system of difference equations:

$$x_{n+1} = A + \frac{y_{n-4}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-4}}{x_n}, \quad n = 0, 1, \ldots \tag{2.17}$$

with $A = 2$, $B = 3$, and the initial conditions $x_{-4} = 0.7$, $x_{-3} = 1.5$, $x_{-2} = 1$, $x_{-1} = 2.1$, $x_0 = 0.5$, $y_{-4} = 2.3$, $y_{-3} = 1$, $y_{-2} = 0.3$, $y_{-1} = 0.2$, $y_0 = 0.9$. Then the unique positive equilibrium $(\bar{x}, \bar{y}) = (3, 4)$ is globally asymptotically stable since $A > 1$ and $B > 1$ (see Theorem 2.3.6), and the solution of system (2.17) is bounded and persists (see Figure 2.3, Theorem 2.3.2). Note that in this example $k = 4$ is even, while in Example 2.5.2, $k = 5$ is odd, but in both cases we had the same conclusion.

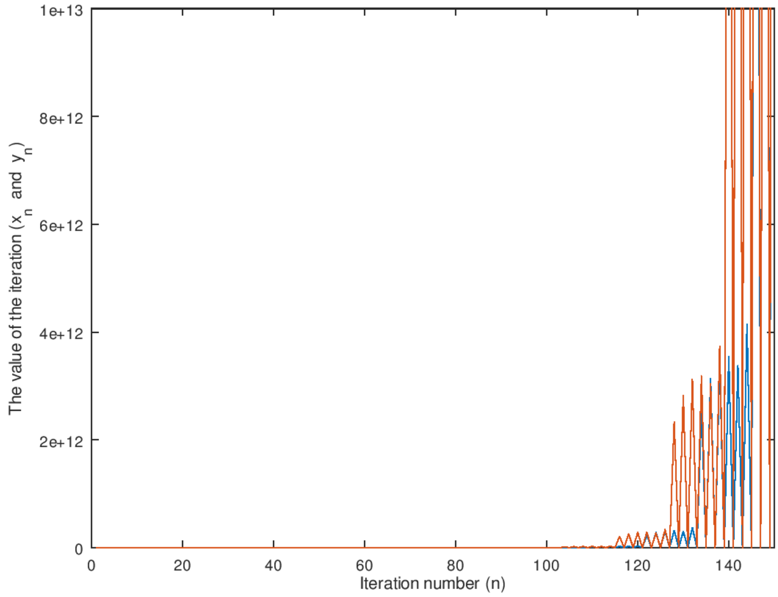

Figure 2.1: The plot of the positive solution of system (2.16) with $A = 0.1$ and $B = 0.9$

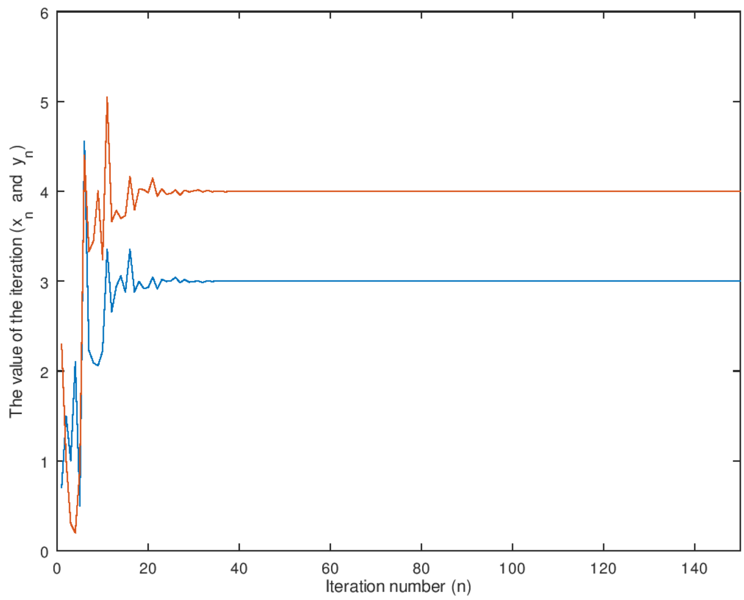

Figure 2.3: The plot of the positive solution of system (2.17) with $A = 2$ and $B = 3$

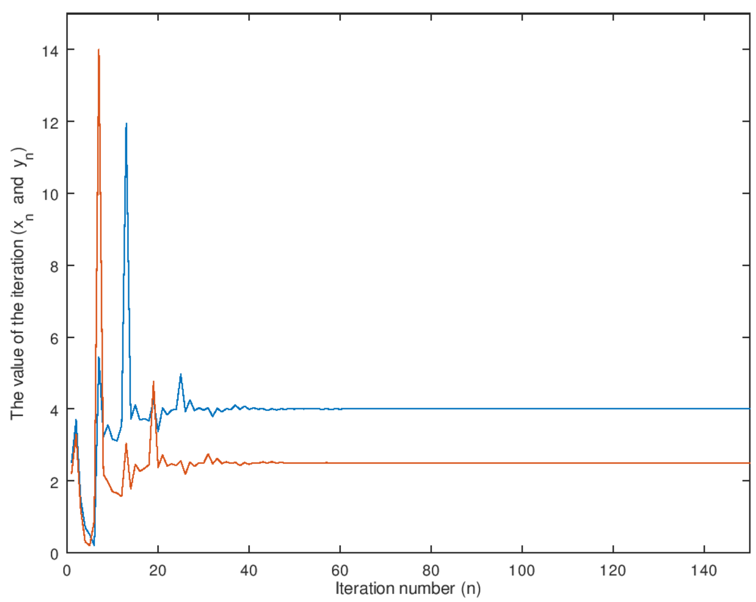

Figure 2.2: The plot of the positive solution of system (2.16) with $A = 3$ and $B = 1.5$

Chapter 3

Dynamics of the System $x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}$

In this chapter, we introduce the symmetrical system:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \ldots \tag{3.1}$$

with parameter $A > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \ldots, 0$ and $k \in \mathbb{Z}^+$. We study the semi-cycles of the positive solutions of system (3.1), we also investigate the dynamical behavior of the solutions of the same system when the parameter $A > 1$, $A = 1$ and $0 < A < 1$. Finally, we provide numerical examples to confirm our results.

The previous system has a unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$. Since $f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ implies $\bar{x} = A + \frac{\bar{y}}{\bar{y}} = A+1$, and $\bar{y} = A + \frac{\bar{x}}{\bar{x}} = A+1$, so $(\bar{x}, \bar{y}) = (A+1, A+1)$.

There are two cases to consider:

Case 1: If the initial conditions x_i, y_i in system (3.1) satisfy the equalities $x_i = y_i$ for $i = -k, -k+1, \ldots, 0$, and $k \in \mathbb{Z}^+$, then

$$x_1 = A + \frac{y_0}{y_{-k}} = A + \frac{x_0}{x_{-k}} = y_1 \quad \text{and} \quad x_2 = A + \frac{y_1}{y_{-k+1}} = A + \frac{x_1}{x_{-k+1}} = y_2$$

by induction, if $x_i = y_i$ for all $i \le m$, then $x_{m+1} = A + \frac{y_m}{y_{m-k}} = A + \frac{x_m}{x_{m-k}} = y_{m+1}$. Hence, $x_n = y_n$ for all $n \ge -k$, thus, system (3.1) reduces to the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-k}} \tag{3.2}$$

which was studied in [1] by Abu-Saris and Devault who showed that every solution of equation (3.2) is bounded and persists, and that the unique positive equilibrium $\bar{x} = 1 + A$ of equation (3.2) is globally asymptotically stable if $A > 1$. They also improved this result for $k = 2$ and 3, and studied the semi-cycles of the nontrivial solutions of equation (3.2).

Case 2: If $x_i \neq y_i$ for some $i \in \{-k, -k+1, \ldots, 0\}, k \in \mathbb{Z}^+$, then this is the case we're about to study in this chapter.

3.1 Semi-cycle Analysis

In this section, we characterize the behavior of positive solutions of system (3.1) about the equilibrium using semi-cycle analysis method.

Theorem 3.1.1. *Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution to system (3.1). Then, either this solution is non-oscillatory solution or it oscillates about the equilibrium*

$(\bar{x}, \bar{y}) = (A+1, A+1)$ *with semi-cycles such that if there exists a semi-cycle with at least* k *terms, then every semi-cycle after that has at least* $k+1$ *terms.*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is a solution to system (3.1), and there exists an integer $n_0 \geq 0$ such that (x_{n_0}, y_{n_0}) is the last term of a semi-cycle that has at least k terms. Then, either

$$\ldots, x_{n_0-k+1}, \ldots, x_{n_0-1}, x_{n_0} < 1 + A \leq x_{n_0+1}$$

and

$$\ldots, y_{n_0-k+1}, \ldots, y_{n_0-1}, y_{n_0} < 1 + A \leq y_{n_0+1}$$

or

$$\ldots, x_{n_0-k+1}, \ldots, x_{n_0-1}, x_{n_0} \geq 1 + A > x_{n_0+1}$$

and

$$\ldots, y_{n_0-k+1}, \ldots, y_{n_0-1}, y_{n_0} \geq 1 + A > y_{n_0+1}$$

Case 1: if $\ldots, x_{n_0-k+1}, \ldots, x_{n_0-1}, x_{n_0} < 1 + A \leq x_{n_0+1}$ and $\ldots, y_{n_0-k+1}, \ldots, y_{n_0-1}, y_{n_0} < 1 + A \leq y_{n_0+1}$, then

$$\begin{aligned}
x_{n_0+2} &= A + \frac{y_{n_0+1}}{y_{n_0-k+1}} > A+1 \text{ and } y_{n_0+2} = A + \frac{x_{n_0+1}}{x_{n_0-k+1}} > A+1 \\
x_{n_0+3} &= A + \frac{y_{n_0+2}}{y_{n_0-k+2}} > A+1 \text{ and } y_{n_0+3} = A + \frac{x_{n_0+2}}{x_{n_0-k+2}} > A+1 \\
&\vdots \\
x_{n_0+k+1} &= A + \frac{y_{n_0+k}}{y_{n_0}} > A+1 \text{ and } y_{n_0+k+1} = A + \frac{x_{n_0+k}}{x_{n_0}} > A+1
\end{aligned}$$

hence, the semi-cycle starting with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume the semi-cycle which starts with (x_{n_0+1}, y_{n_0+1}) has exactly $k+1$ terms, then the following semi-cycle will start with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that $x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+k+1} \geq 1+A > x_{n_0+k+2}$ and $y_{n_0+1}, y_{n_0+2}, \ldots, y_{n_0+k+1} \geq 1+A > y_{n_0+k+2}$, then for $i = 1, 3, \ldots, k$

$$x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} < A+1 \text{ and } y_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{x_{n_0+k+1+i}} < A+1$$

from here, it's clear that every semi-cycle after this point must have at least $k+1$ terms.

Case 2: if $\ldots, x_{n_0-k+1}, \ldots, x_{n_0-1}, x_{n_0} \geq 1+A > x_{n_0+1}$ and $\ldots, y_{n_0-k+1}, \ldots, y_{n_0-1}, y_{n_0} \geq 1+A > y_{n_0+1}$, then for all $i = 2, 3, \ldots, k+1$

$$x_{n_0+i} = A + \frac{y_{n_0-1+i}}{y_{n_0-k-1+i}} < A+1 \text{ and } y_{n_0+i} = A + \frac{x_{n_0-1+i}}{x_{n_0-k-1+i}} < A+1$$

hence, the semi-cycle starting with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume this semi-cycle has exactly $k+1$ terms, then the following semi-cycle will start with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that $x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+k+1} < 1+A \leq x_{n_0+k+2}$ and $y_{n_0+1}, y_{n_0+2}, \ldots, y_{n_0+k+1} < 1+A \leq y_{n_0+k+2}$, then for $i = 1, 3, \ldots, k$

$$x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} > A+1 \text{ and } y_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{x_{n_0+1+i}} > A+1$$

now, it's clear that every semi-cycle after this point must have at least $k+1$ terms. Hence, if there exists a semi-cycle with at least k terms, then every semi-cycle after that has at least $k+1$ terms. The proof is complete. $\square$

Theorem 3.1.2. *System (3.1) has no nontrivial periodic solutions of period k (not necessarily prime period k).*

Proof. Assume system (3.1) has a k-periodic solution. Then, $(x_{n-k}, y_{n-k}) =$

(x_n, y_n) for all $n \geq 0$, and so

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} = A + 1, \text{ and } y_{n+1} = A + \frac{x_n}{x_{n-k}} = A + 1, \quad \text{forall } n \geq 0$$

Thus, the solution $(x_n, y_n) = (A + 1, A + 1)$ is the equilibrium solution of (3.1). □

Theorem 3.1.3. *Any increasing solution to system (3.1) is non-oscillatory positive (the infinite semi-cycle in the solution is a positive semi-cycle).*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is an increasing non-oscillatory solution to system (3.1). Then, either $A + 1 \leq x_1$ and $A + 1 \leq y_1$ or $x_1 < A + 1$ and $y_1 < A + 1$.

Case 1: if $A + 1 \leq x_1$ and $A + 1 \leq y_1$, since the solution is increasing then $A + 1 \leq x_1 \leq x_2 \leq x_3 \leq \ldots$ and $A + 1 \leq y_1 \leq y_2 \leq y_3 \leq \ldots$, so the solution has an infinite positive semi-cycle. We also can see that as soon as the solution enters a positive semi-cycle, it remains in this semi-cycle.

Case 2: if $x_1 < A+1$ and $y_1 < A+1$, then we claim that the semi-cycle containing (x_1, y_1) ends with (x_i, y_i) such that $1 \leq i \leq k + 1$. If $i = k + 2$, then

$$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A + 1 \text{ and } y_{k+2} = A + \frac{x_{k+1}}{x_1} < A + 1$$

imply that

$$y_{k+1} < y_1 \text{ and } x_{k+1} < x_1 \text{ but } k + 1 > 1$$

which contradicts the fact that the solution is increasing, so any increasing solution of system is non-oscillatory positive. Moreover, if the increasing solution has a negative semi-cycle, then this semi-cycle can have at most $2k+2$ terms. □

Theorem 3.1.4. *System (3.1) has no non-oscillatory negative solutions (has no infinite negative semi-cycle).*

Proof. On the contrary, assume system (3.1) has a non-oscillatory solution say $\{x_n, y_n\}_{n=-k}^{\infty}$ which has an infinite negative semi-cycle, and assume this semi-cycle starts with (x_N, y_N), where $N \geq -k$. Then for all $n \geq N$, $(x_n, y_n) < (A+1, A+1)$, hence

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} < A+1 \quad \text{implies} \quad y_n < y_{n-k} \;\text{ for }\; n \geq \max\{1, N-1\}$$

and

$$y_{n+1} = A + \frac{x_n}{x_{n-k}} < A+1 \quad \text{implies} \quad x_n < x_{n-k} \;\text{ for }\; n \geq \max\{1, N-1\}$$

so for all $n \geq \max\{1, N\}$

$$A < \cdots < x_{n+k} < x_n < x_{n-k} < A+1 \;\text{ and}$$
$$A < \cdots < y_{n+k} < y_n < y_{n-k} < A+1$$

which means that $\{x_n\}, \{y_n\}$ have k subsequences

$$\{x_{nk}\}, \{x_{nk+1}\}, ..., \{x_{nk+(k-1)}\} \;\text{ and }\; \{y_{nk}\}, \{y_{nk+1}\}, ..., \{y_{nk+(k-1)}\}$$

each subsequence is decreasing and bounded from below, so each one of them is convergent, so for all $i = 0, 1, ..., k-1$ there exist α_i, β_i such that

$$\lim_{n\to\infty} x_{nk+i} = \alpha_i \;\text{ and }\; \lim_{n\to\infty} y_{nk+i} = \beta_i$$

Thus,

$$(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{k-1}, \beta_{k-1})$$

is a k-periodic solution of system (3.1), which contradicts Theorem 3.1.2 unless the solution is the trivial solution. Hence, the solution converges to the equilibrium, which is a contradiction, because the solution is diverging from the equilibrium. Hence, system (3.1) has no non-oscillatory negative solutions. □

Theorem 3.1.5. *System (3.1) has no decreasing non-oscillatory solutions.*

Proof. Assume system (3.1) has a decreasing non-oscillatory solution say $\{x_n, y_n\}_{n=-k}^{\infty}$. As in proof of Theorem 3.1.3, the solution is either of the form

$$\cdots \leq x_3 \leq x_2 \leq x_1 \leq A+1 \text{ and } \cdots \leq y_3 \leq y_2 \leq y_1 \leq A+1$$

or there exists a positive integer $n_0 \geq k+1$, such that

$$\cdots \leq x_{n_0+2} \leq x_{n_0+1} \leq A+1 \leq x_{n_0} \leq x_{n_0-1} \ldots$$

and

$$\cdots \leq y_{n_0+2} \leq y_{n_0+1} \leq A+1 \leq y_{n_0} \leq y_{n_0-1} \ldots$$

where the positive semi-cycle ending with (x_{n_0}, y_{n_0}) can have at most $2k+2$ terms. In both cases, the solution has an infinite negative semi-cycle which contradicts Theorem 3.1.4. Hence, system (3.1) has no decreasing non-oscillatory solutions. □

Theorem 3.1.6. *Consider system (3.1). If k is even, then the following statements hold:*

(a) *Every semi-cycle has length at most* $2k+1$.

(b) *The extreme term in a semi-cycle occurs in the first* $k+2$ *terms of the semi-cycle.*

(c) *Every solution oscillates about* $(\bar{x}, \bar{y}) = (A+1, A+1)$.

Proof. Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution of system (3.1). In the case of a negative semi-cycle, let (x_N, y_N) be the first term in a negative semi-cycle, and suppose this semi-cycle is of length $2k+1$. Then

$$x_N, x_{N+1}, \ldots, x_{N+2k} < A+1 \text{ and } y_N, y_{N+1}, \ldots, y_{N+2k} < A+1$$

for $i = 1, 2, \ldots, k-1$, we have

$$x_{N+k+i+1} = A + \frac{y_{N+k+i}}{y_{N+i}} > A + \frac{y_{N+k+i}}{A+1} > y_{N+k+i}, \quad \text{since } y_{N+k+i} < A+1$$

and

$$y_{N+k+i} = A + \frac{x_{N+k+i-1}}{x_{N+i-1}} > A + \frac{x_{N+k+i-1}}{A+1} > x_{N+k+i-1}$$

which imply that

$$x_{N+k} < y_{N+k+1} < x_{N+k+2} < y_{N+k+3} < x_{N+k+4} < \cdots < x_{N+2k}$$

and

$$y_{N+k} < x_{N+k+1} < y_{N+k+2} < x_{N+k+3} < y_{N+k+4} < \cdots < y_{N+2k}$$

so

$$x_{N+k} < x_{N+2k} \quad \text{and} \quad y_{N+k} < y_{N+2k}$$

now, for

$$x_{N+2k+1} = A + \frac{y_{N+2k}}{y_{N+k}} > A+1 \quad \text{and} \quad y_{N+2k+1} = A + \frac{x_{N+2k}}{x_{N+k}} > A+1$$

and so a negative semi-cycle has at most $2k+1$ terms. The case of a positive semi-cycle is similar to the previous case. Let (x_N, y_N) be the first term in a positive semi-cycle, and suppose this semi-cycle has $2k+1$ terms. Then

$$x_N, x_{N+1}, \ldots, x_{N+2k} \geq A+1 \;\text{ and }\; y_N, y_{N+1}, \ldots, y_{N+2k} \geq A+1$$

for $i = 1, 2, \ldots, k-1$, we have

$$x_{N+k+i+1} = A + \frac{y_{N+k+i}}{y_{N+i}} \leq A + \frac{y_{N+k+i}}{A+1} \leq y_{N+k+i}, \quad \text{since } y_{N+k+i} \geq A+1$$

and

$$y_{N+k+i} = A + \frac{x_{N+k+i-1}}{x_{N+i-1}} \leq A + \frac{x_{N+k+i-1}}{A+1} \leq x_{N+k+i-1}$$

which imply that

$$x_{N+k} \geq y_{N+k+1} \geq x_{N+k+2} \geq y_{N+k+3} \geq x_{N+k+4} \geq \cdots \geq x_{N+2k}$$

and

$$y_{N+k} \geq x_{N+k+1} \geq y_{N+k+2} \geq x_{N+k+3} \geq y_{N+k+4} \geq \cdots \geq y_{N+2k}$$

now, for

$$x_{N+2k+1} = A + \frac{y_{N+2k}}{y_{N+k}} \leq A+1 \quad \text{and} \quad y_{N+2k+1} = A + \frac{x_{N+2k}}{x_{N+k}} \leq A+1$$

and so a semi-cycle has at most $2k+1$ terms. From this proof, it is obvious that the extreme term in a semi-cycle occurs in the first $k+2$ terms. Since every semi-cycle is of length at most $2k+1$, this implies that the solution oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$. □

3.2 The Case $0 < A < 1$

In this section, we study the asymptotic behavior of system (3.1) when $0 < A < 1$, we also prove that when $0 < A < 1$, system (3.1) can have unbounded solution given some certain conditions.

Theorem 3.2.1. *Assume that $0 < A < 1$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ is an arbitrary positive solution of (3.1). Then the following statements are true:*

1. *If k is odd and $0 < x_{2m-1} < 1$, $x_{2m} > \frac{1}{1-A}$, $y_{2m-1} > \frac{1}{1-A}$, $0 < y_{2m} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then*

$$\lim_{n\to\infty} x_{2n} = \infty, \lim_{n\to\infty} y_{2n+1} = \infty, \lim_{n\to\infty} x_{2n+1} = A, \lim_{n\to\infty} y_{2n} = A$$

2. *If k is odd and $0 < x_{2m} < 1$, $x_{2m-1} > \frac{1}{1-A}$, $y_{2m} > \frac{1}{1-A}$, $0 < y_{2m-1} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then*

$$\lim_{n\to\infty} x_{2n+1} = \infty, \lim_{n\to\infty} y_{2n} = \infty, \lim_{n\to\infty} x_{2n} = A, \lim_{n\to\infty} y_{2n+1} = A$$

Proof. 1. If k is odd and $0 < x_{2m-1} < 1$, $x_{2m} > \frac{1}{1-A}$, $y_{2m-1} > \frac{1}{1-A}$,

$0 < y_{2m} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \dots, 0$, then it is clear that

$$\begin{aligned}
0 < x_1 =& A + \frac{y_0}{y_{-k}} < A + \frac{1}{y_{-k}} < A + 1 - A = 1 \\
y_1 =& A + \frac{x_0}{x_{-k}} > A + x_0 > x_0 > \frac{1}{1-A} \\
x_2 =& A + \frac{y_1}{y_{-k+1}} > A + y_1 > y_1 > \frac{1}{1-A} \\
0 < y_2 =& A + \frac{x_1}{x_{-k+1}} < A + \frac{1}{x_{-k+1}} < A + 1 - A = 1
\end{aligned}$$

By induction, we get that for all $n = 1, 2, \dots$

$$0 < x_{2n-1} < 1, x_{2n} > \frac{1}{1-A}, y_{2n-1} > \frac{1}{1-A}, 0 < y_{2n} < 1$$

so for $l \geq 1$

$$\begin{aligned}
x_{2l} &= A + \frac{y_{2l-1}}{y_{2l-(k+1)}} > A + y_{2l-1} = 2A + \frac{x_{2l-2}}{x_{2l-k-2}} > 2A + x_{2l-2} \\
x_{4l} &= A + \frac{y_{4l-1}}{y_{4l-(k+1)}} > A + y_{4l-1} = 2A + \frac{x_{4l-2}}{x_{4l-k-2}} > 2A + x_{4l-2} \\
&= 3A + \frac{y_{4l-3}}{y_{4l-k-3}} > 3A + y_{4l-3} = 4A + \frac{x_{4l-4}}{x_{4l-k-4}} > 4A + x_{4l-4}
\end{aligned}$$

also

$$x_{6l} > 6A + x_{6l-6}$$

so for all $r = 1, 2, \dots$

$$x_{2rl} > 2rA + x_{2rl-2r}$$

if $n = rl$, then as $r \to \infty, n \to \infty$ and $\lim_{n\to\infty} x_{2n} = \infty$. Considering (3.1) and taking the limits on both sides of the equation

$$y_{2n+1} = A + \frac{x_{2n}}{x_{2n-k}}$$

we get $\lim_{n\to\infty} y_{2n+1} = \infty$ since $0 < x_{2n-k} < 1$ for all $n = 0, 1, \ldots$. Now, take the limits on both sides of the equation

$$x_{2n+1} = A + \frac{y_{2n}}{y_{2n-k}}$$

we obtain $\lim_{n\to\infty} x_{2n+1} = A$ since $0 < y_{2n} < 1$ for all n. Similarly, take the limits on both sides of the equation

$$y_{2n+2} = A + \frac{x_{2n+1}}{x_{2n-k+1}}$$

to get $\lim_{n\to\infty} y_{2n+1} = A$. Hence, we complete the proof of 1.

2. If k is odd and $0 < x_{2m} < 1$, $x_{2m-1} > \frac{1}{1-A}$, $y_{2m} > \frac{1}{1-A}$, $0 < y_{2m-1} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then

$$\begin{aligned} x_1 =& A + \frac{y_0}{y_{-k}} > A + y_0 > y_0 > \frac{1}{1-A} \\ 0 < y_1 =& A + \frac{x_0}{x_{-k}} < A + \frac{1}{x_{-k}} << A + 1 - A = 1 \\ 0 < x_2 =& A + \frac{y_1}{y_{-k+1}} < A + \frac{1}{y_{-k+1}} < A + 1 - A = 1 \\ y_2 =& A + \frac{x_1}{x_{-k+1}} > A + x_1 > x_1 > \frac{1}{1-A} \end{aligned}$$

By induction, we have for all $n = 1, 2, \ldots$

$$0 < x_{2n} < 1, x_{2n-1} > \frac{1}{1-A}, y_{2n} > \frac{1}{1-A}, 0 < y_{2n-1} < 1$$

so for $l \geq 1$

$$\begin{aligned} x_{2l+1} &= A + \frac{y_{2l}}{y_{2l-k}} > A + y_{2l} = 2A + \frac{x_{2l-1}}{x_{2l-k-1}} > 2A + x_{2l-1} \\ x_{4l+1} &= A + \frac{y_{4l}}{y_{4l-k}} > A + y_{4l} = 2A + \frac{x_{4l-1}}{x_{4l-k-1}} > 2A + x_{4l-1} \\ &= 3A + \frac{y_{4l-2}}{y_{4l-k-2}} > 3A + y_{4l-2} = 4A + \frac{x_{4l-3}}{x_{4l-k-3}} > 4A + x_{4l-3} \end{aligned}$$

similarly, $x_{6l+1} > 6A + x_{6l-5}$. So for all $r = 1, 2, \ldots$

$$x_{2rl+1} > 2rA + x_{2rl-(2r-1)}$$

if $n = rl$, then as $r \to \infty, n \to \infty$ and $\lim_{n\to\infty} x_{2n+1} = \infty$. Considering (3.1) and taking the limits on both sides of the equation

$$y_{2n+2} = A + \frac{x_{2n+1}}{x_{2n-k+1}}$$

we get $\lim_{n\to\infty} y_{2n} = \infty$ since $0 < x_{2n-k+1} < 1$ for all $n = 0, 1, \ldots$. Now, take the limits on both sides of the equation

$$y_{2n+1} = A + \frac{x_{2n}}{x_{2n-k}}$$

we obtain $\lim_{n\to\infty} y_{2n+1} = A$ since $0 < x_{2n} < 1$ for all n. Similarly, take the limits on both sides of the equation

$$x_{2n+2} = A + \frac{y_{2n+1}}{y_{2n-k+1}}$$

to get $\lim_{n\to\infty} x_{2n} = A$, which completes the proof.

$\square$

3.3 The Case $A = 1$

In this section, we study the boundedness , persistence and periodicity of positive solutions of system (3.1) when $A = 1$.

Theorem 3.3.1. *If $A = 1$, then every positive solution of system (3.1) is bounded and persists.*

Proof. Assume $A = 1$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of system (3.1). Then $x_n, y_n > A = 1$ for all $n > 0$, so we can choose a real number L to be close enough to 1 such that $1 < L < \frac{L}{L-1}$ and $x_i, y_i \in [L, \frac{L}{L-1}]$ for all $i = 1, \ldots, k+1$. Now from (3.1), $x_{k+2} = 1 + \frac{y_{k+1}}{y_1}$ and $y_{k+2} = 1 + \frac{x_{k+1}}{x_1}$, then

$$L = 1 + \frac{L}{L/(L-1)} \leq x_{k+2}, y_{k+2} \leq 1 + \frac{L/(L-1)}{L} = \frac{L-1+1}{L-1} = \frac{L}{L-1}$$

by induction, $x_i, y_i \in [L, \frac{L}{L-1}]$ for all $i = 1, 2, ...$ The proof is complete. □

Theorem 3.3.2. *Suppose $A = 1$, $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of system (3.1). Then*

$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n$$

$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n$$

Proof. Let

$$l_1 = \liminf_{n\to\infty} x_n, \quad l_2 = \liminf_{n\to\infty} y_n$$

$$u_1 = \limsup_{n\to\infty} x_n, \quad u_2 = \limsup_{n\to\infty} y_n$$

Clearly, $1 < l_1 \leq u_1$ and $1 < l_2 \leq u_2$. Now, since $x_{n+1} = 1 + \frac{y_n}{y_{n-k}}$, $y_{n+1} = 1 + \frac{x_n}{x_{n-k}}$, $l_1 \leq x_i \leq u_1$ and $l_2 \leq y_i \leq u_2$ for all i, so

$$l_1 \geq 1 + \frac{l_2}{u_2}, \quad l_2 \geq 1 + \frac{l_1}{u_1}, \quad u_1 \leq 1 + \frac{u_2}{l_2}, \quad u_2 \leq 1 + \frac{u_1}{l_1}$$

so

$$l_1 u_2 \geq l_2 + u_2, \quad l_2 u_1 \geq l_1 + u_1, \quad u_1 l_2 \leq l_2 + u_2, \quad u_2 l_1 \leq l_1 + u_1$$

so

$$l_2 u_1 \leq l_2 + u_2 \leq l_1 u_2 \leq l_1 + u_1 \leq l_2 u_1$$

so

$$l_2 + u_2 = l_2 u_1 \tag{3.3}$$

$$l_1 u_2 = l_2 u_1 \tag{3.4}$$

$$l_1 + u_1 = l_2 u_1 \tag{3.5}$$

from (3.4), we can get

$$\frac{l_1}{l_2} = \frac{u_1}{u_2} \tag{3.6}$$

divide (3.5) by l_2 to get

$$\frac{l_1}{l_2} + \frac{u_1}{l_2} = u_1 \tag{3.7}$$

substitute (3.6) in (3.7) to get

$$\frac{u_1}{u_2} + \frac{u_1}{l_2} = u_1$$

which implies

$$\frac{1}{u_2} + \frac{1}{l_2} = 1, \text{ and } l_2 = \frac{u_2}{u_2 - 1} \tag{3.8}$$

substitute (3.8) in (3.3) to get

$$u_2 + \frac{u_2}{u_2 - 1} = u_1 \cdot \frac{u_2}{u_2 - 1}$$

which implies

$$1 + \frac{1}{u_2 - 1} = \frac{u_1}{u_2 - 1}$$

so, we can get

$$\frac{u_2}{u_2 - 1} = \frac{u_1}{u_2 - 1}$$

Hence, $u_1 = u_2$, and from (3.4) we can get $l_1 = l_2$. The result then follows. □

Theorem 3.3.3. *Suppose $A = 1$.*

1. *If k is odd, then every positive solution of system (3.1) with prime period two takes the form*

$$\ldots, (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), \ldots \quad \textit{with } 1 < a \neq 2$$

2. *If k is even, there do not exist positive nontrivial solution of system (3.1) with prime period two.*

Proof. 1. Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a positive two periodic solution. Then there exist $a, b, c, d \in \mathbb{R}^+$, all are greater than 1, such that for all $n \geq 0$, $x_{2n-k} = a$, $y_{2n-k} = b$, $x_{2n+1-k} = c$, $y_{2n+1-k} = d$, that is, the solution is $\ldots, (a, b), (c, d), (a, b), (c, d), \ldots$

Case 1: if $a = c$, then $x_{2n-k} = x_{2n+1-k}$, so $x_{-k} = x_{-k+1} = x_{-k+2} = x_{-k+3} = \ldots$, so $b = d$, which implies that the solution is not two periodic which is not the case.

Case 2: let $a \neq c$. Then $b \neq d$, so using the previous theorem, we can get

$$\min\{a, c\} = \liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = \min\{b, d\}$$

and

$$\max\{a, c\} = \limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = \max\{b, d\}$$

hence, we have the following cases:

(a) if $a < c$ and $b < d$, then $a = b$ and $c = d$, then the solution is of the form: $\dots, (a, a), (c, c), (a, a), (c, c), \dots$

(b) if $a < c$ and $b > d$, then $a = d$ and $c = b$, then the solution is of the form: $\dots, (a, c), (c, a), (a, c), (c, a), \dots$

(c) if $a > c$ and $b < d$, then we get case (b).

(d) if $a > c$ and $b > d$, then we get case (a).

In **(a)**, $x_{2n-k} = a,\ y_{2n-k} = a,\ x_{2n+1-k} = c,\ y_{2n+1-k} = c$. But $x_1 = 1 + \frac{y_0}{y_{-k}}$, so $a = 1 + \frac{c}{a}$, which implies $a^2 = c + a$, also $x_2 = 1 + \frac{y_1}{y_{-k+1}}$, so $c = 1 + \frac{a}{c}$, which implies $c^2 = c + a$ and hence, $a^2 = c^2$ but $a \neq c$, so $c = -a$. Which is not the case, because we only consider the positive solutions.

In **(b)**, $x_{2n-k} = a,\ y_{2n-k} = c,\ x_{2n+1-k} = c,\ y_{2n+1-k} = a$. But $x_1 = 1 + \frac{y_0}{y_{-k}}$, so $a = 1 + \frac{a}{c}$, which implies $ac = c + a$, and hence, $c = \frac{a}{a-1}$. Then the solution is $\dots, (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), (a, \frac{a}{a-1}), (\frac{a}{a-1}, a), \dots$.

2. Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a positive two periodic solution and k is even, if $x_{2n-k} = a,\ y_{2n-k} = b,\ x_{2n+1-k} = c,\ y_{2n+1-k} = d$, then

Case 1: if $a = c$, then $b = d$ so the solution is not two periodic which is not the case.

Case 2: if $a \neq c$, then $b \neq d$, so

$$\min\{a, c\} = \liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = \min\{b, d\}$$

and

$$\max\{a, c\} = \limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = \max\{b, d\}$$

The same as before:

(a) if $a < c$ and $b < d$, then $a = b$ and $c = d$, then the solution is of the form: $\ldots, (a, a), (c, c), (a, a), (c, c), ...$

(b) if $a < c$ and $b > d$, then $a = d$ and $c = b$, then the solution is of the form: $\ldots, (a, c), (c, a), (a, c), (c, a), ...$

(c) if $a > c$ and $b < d$, then we get case (b).

(d) if $a > c$ and $b > d$, then we get case (a).

In **(a)**, $x_{2n-k} = a$, $y_{2n-k} = a$, $x_{2n+1-k} = c$, $y_{2n+1-k} = c$. But $x_1 = 1+\frac{y_0}{y_{-k}}$, so $c = 1+\frac{a}{a}$, which implies $c = 2$, also $x_2 = 1+\frac{y_1}{y_{-k+1}}$, so $a = 1+\frac{c}{c}$, which implies $a = 2$. Then the solution is $\ldots, (2, 2), (2, 2), \ldots,$ which is not two periodic .

In **(b)**, $x_{2n-k} = a$, $y_{2n-k} = c$, $x_{2n+1-k} = c$, $y_{2n+1-k} = a$. But $x_1 = 1 + \frac{y_0}{y_{-k}}$, so $a = 1 + \frac{c}{c}$, which implies $a = 2$, also $x_2 = 1 + \frac{y_1}{y_{-k+1}}$, so $c = 1+\frac{2}{a}$, which implies $c = 2$. Then the solution is $\ldots, (2, 2), (2, 2), \ldots,$ which is not two periodic. So when k is even, there is no nontrivial two periodic solution.

□

3.4 The Case $A > 1$

In this section, we study the boundedness and persistence of the positive solutions of system (3.1) when $A > 1$, and we show that the unique positive equilibrium is a globally asymptotically stable.

Lemma 3.4.1. *Given v_k, v_{k+1}. Then the solution of the second order linear difference equation*

$$v_{n+2} = av_n + b, \quad n \geq k, \quad a \neq 1$$

is

$$v_{k+2l} = \left(v_k + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$
$$v_{k+2l+1} = \left(v_{k+1} + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$

for all $l \geq 0$.

Proof.

when $n = k$, $v_{k+2} = av_k + b$

when $n = k+1$, $v_{k+3} = av_{k+1} + b$

when $n = k+2$, $v_{k+4} = av_{k+2} + b = a^2 v_k + ab + b$

when $n = k+3$, $v_{k+5} = av_{k+3} + b = a^2 v_{k+1} + ab + b$

when $n = k+4$, $v_{k+6} = av_{k+4} + b = a^3 v_k + a^2 b + ab + b$

when $n = k+5$, $v_{k+7} = av_{k+5} + b = a^3 v_{k+1} + a^2 b + ab + b$

hence, for all $l \geq 0$

$$v_{k+2l} = a^l v_k + b(a^{l-1} + a^{l-2} + \cdots + 1) = \left(v_k + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$
$$v_{k+2l+1} = a^l v_{k+1} + b(a^{l-1} + a^{l-2} + \cdots + 1) = \left(v_{k+1} + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$

this completes the proof. □

Theorem 3.4.2. *Suppose $A > 1$. Then every positive solution of system (3.1) is bounded and persists. In fact, for all $l \geq 0$,*

$$A < x_{k+2l} \leq \left(x_k + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

and

$$A < x_{k+2l+1} \leq \left(x_{k+1} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

similarly,

$$A < y_{k+2l} \leq \left(y_k + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

and

$$A < y_{k+2l+1} \leq \left(x_{k+1} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

Proof. Assume $A > 1$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of system (3.1). Since $x_n > 0$ and $y_n > 0$ for all $n \geq -k$, (3.1) implies that

$$x_n, y_n > A \text{ for all } n \geq 1 \tag{3.9}$$

use (3.1) and (3.9) to get that for all $n \geq k+2$

$$x_n = A + \frac{y_{n-1}}{y_{n-k-1}} < A + \frac{1}{A}y_{n-1}, \ \ y_n < A + \frac{1}{A}x_{n-1} \tag{3.10}$$

Let $\{v_n, w_n\}$ be the solution of the following system

$$v_n = A + \frac{1}{A}w_{n-1}, \ \ w_n = A + \frac{1}{A}v_{n-1} \text{ for all } n \geq k+2 \tag{3.11}$$

such that

$$v_i = x_i, \ \ w_i = y_i, \ \ i = 1, 2, \ldots, k+1 \tag{3.12}$$

now, we use induction to prove that

$$x_n < v_n, \ \ y_n < w_n, \ \ n \geq k+2 \tag{3.13}$$

Suppose that (3.13) is true for $n = m \geq k+2$. Then, from (3.10), we get

$$\begin{aligned} x_{m+1} < A + \frac{1}{A}y_m < A + \frac{1}{A}w_m = v_{m+1} \\ y_{m+1} < A + \frac{1}{A}x_m < A + \frac{1}{A}v_m = w_{m+1} \end{aligned} \tag{3.14}$$

Therefore, (3.13) is true. From (3.11) and (3.12), we have

$$v_{n+2} = \frac{1}{A^2}v_n + A + 1, \ \ w_{n+2} = \frac{1}{A^2}w_n + A + 1, \ \ n \geq k \tag{3.15}$$

for simplicity, let $a = \frac{1}{A^2}$ and $b = A + 1$. Then (3.15) becomes

$$v_{n+2} = av_n + b, \quad w_{n+2} = aw_n + b, \quad n \geq k$$

Now, using Lemma 3.4.1, for all $l \geq 0$

$$v_{k+2l} = a^l x_k + b(a^{l-1} + a^{l-2} + \cdots + 1) = \left(x_k + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$
$$v_{k+2l+1} = a^l x_{k+1} + b(a^{l-1} + a^{l-2} + \cdots + 1) = \left(x_{k+1} + \frac{b}{a-1}\right) a^l + \frac{b}{1-a}$$

since $A > 1$, and $a = \frac{1}{A^2}$, $b = A + 1$. Then for all $l \geq 0$

$$\begin{aligned} v_{k+2l} &= \left(x_k + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1} \\ v_{k+2l+1} &= \left(x_{k+1} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1} \end{aligned} \tag{3.16}$$

Then, from (3.9), (3.13), and (3.16), for all $l \geq 0$

$$A < x_{k+2l} \leq \left(x_k + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

and

$$A < x_{k+2l+1} \leq \left(x_{k+1} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

Similarly, we get

$$A < y_{k+2l} \leq \left(y_k + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

and

$$A < y_{k+2l+1} \leq \left(x_{k+1} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{2l} + \frac{A^2}{A-1}$$

The proof is complete. □

Theorem 3.4.3. *Suppose $A > 1$. Then every positive solution of system (3.1) converges to the equilibrium $(A+1, A+1)$ as $n \to \infty$.*

Proof. Let

$$l_1 = \liminf_{n\to\infty} x_n, \quad l_2 = \liminf_{n\to\infty} y_n$$
$$u_1 = \limsup_{n\to\infty} x_n, \quad u_2 = \limsup_{n\to\infty} y_n$$

Clearly, $1 < l_1 \le u_1$ and $1 < l_2 \le u_2$. Now, since $x_{n+1} = 1 + \frac{y_n}{y_{n-k}}$ and $y_{n+1} = 1 + \frac{x_n}{x_{n-k}}$, so

$$l_1 \ge A + \frac{l_2}{u_2}, \quad l_2 \ge A + \frac{l_1}{u_1}, \quad u_1 \le A + \frac{u_2}{l_2}, \quad u_2 \le A + \frac{u_1}{l_1}$$

so

$$l_1 u_2 \ge l_2 + A u_2 \tag{3.17}$$

$$l_2 u_1 \ge l_1 + A u_1 \tag{3.18}$$

$$u_1 l_2 \le A l_2 + u_2 \tag{3.19}$$

$$u_2 l_1 \le A l_1 + u_1 \tag{3.20}$$

so

$$A u_1 + l_1 \le l_2 u_1 \le A l_2 + u_2 \tag{3.21}$$

and

$$A u_2 + l_2 \le l_1 u_2 \le A l_1 + u_1 \tag{3.22}$$

from (3.21) and (3.22) we get

$$A u_1 + l_1 + A u_2 + l_2 \le A l_2 + u_2 + A l_1 + u_1$$

which implies

$$A u_1 + l_1 - A l_1 - u_1 \le A l_2 + u_2 - A u_2 - l_2$$

so

$$A(u_1 - l_1 - l_2 + u_2) + (l_1 - u_1 - u_2 + l_2) \leq 0$$

so

$$(A-1)(u_1 - l_1 + u_2 - l_2) \leq 0$$

but $A > 1$ so $A - 1 > 0$, hence

$$u_1 - l_1 + u_2 - l_2 \leq 0$$

but both $u_1 - l_1, u_2 - l_2 \geq 0$, so $u_1 - l_1 + u_2 - l_2 \geq 0$. Hence,

$$u_1 - l_1 + u_2 - l_2 = 0 \text{ iff } u_1 - l_1 = 0 \text{ and } u_2 - l_2 = 0 \text{ iff } u_1 = l_1 \text{ and } u_2 = l_2$$

Now back to (3.17), (3.18), (3.19), (3.20).

$$\text{from (3.17) } l_1 l_2 \geq A l_2 + l_2, \text{ so } l_1 \geq A + 1$$

and

$$\text{from (3.19) } l_2 l_1 \leq A l_2 + l_2, \text{ so } l_1 \leq A + 1$$

so

$$l_1 = A + 1, \text{ so } \lim_{n\to\infty} x_n = l_1 = u_1 = A + 1$$

Similarly, use (3.18) and (3.20) to get

$$l_2 = A + 1, \text{ so } \lim_{n\to\infty} y_n = l_2 = u_2 = A + 1$$

which completes the proof. □

Theorem 3.4.4. *If $A > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A + 1, A + 1)$ of system (3.1) is locally asymptotically stable.*

Proof. System (3.1) can be formulated as the system of first order recurrence equations (2.14). If $Z_n = (w_n^{(1)}, w_n^{(2)}, \ldots, w_n^{(k+1)}, v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(k+1)})^T$, then the linearized equation of system (3.1) associated with (2.14) about the equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is

$$Z_{n+1} = JZ_n$$

where

$$Z_{n+1} = \begin{pmatrix} w_{n+1}^{(1)} \\ w_{n+1}^{(2)} \\ \vdots \\ w_{n+1}^{(k+1)} \\ v_{n+1}^{(1)} \\ v_{n+1}^{(2)} \\ \vdots \\ v_{n+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} A + \frac{v_n^{(1)}}{v_n^{(k+1)}} \\ w_n^{(1)} \\ \vdots \\ w_n^{(k)} \\ A + \frac{w_n^{(1)}}{w_n^{(k+1)}} \\ v_n^{(1)} \\ \vdots \\ v_n^{(k)} \end{pmatrix}$$

and the Jacobian matrix J is of the form:

$$\begin{aligned} & J_{(2k+2)\times(2k+2)} \\ = \quad & \left(D_{w_n^{(1)}} Z_{n+1} \quad \ldots \quad D_{w_n^{(k+1)}} Z_{n+1} \quad D_{v_n^{(1)}} Z_{n+1} \quad \ldots \quad D_{v_n^{(k+1)}} Z_{n+1} \right) \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{1}{A+1} & 0 & \dots & 0 & \frac{-1}{A+1} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{A+1} & 0 & \dots & 0 & \frac{-1}{A+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}$$

Let $\lambda_1, \lambda_2, ..., \lambda_{2k+2}$ be the eigenvalues of J. Define $D = diag(d_1, d_2, ..., d_{2k+2})$ be a diagonal matrix such that

$$d_1 = d_{k+2} = 1, \;\; d_m = d_{k+1+m} = 1 - m\varepsilon, \;\; m = 2, 3, ..., k+1$$

choose $\varepsilon > 0$ such that $0 < \varepsilon < \frac{A-1}{(A+1)(k+1)}$. Now,

$$D_{(2k+2)\times(2k+2)} = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-2\varepsilon & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1-(k+1)\varepsilon & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1-2\varepsilon & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1-(k+1)\varepsilon \end{pmatrix}$$

so for all $m = 2, 3, \dots, k+1$,

$$1 - m\varepsilon \geq 1-(k+1)\varepsilon > 1 - \frac{(k+1)(A-1)}{(k+1)(A+1)} = \frac{A+1-A+1}{A+1} = \frac{2}{A+1} > 0$$

so for all m, $1 - m\varepsilon > 0$, hence D is invertible. Now,

$$DJD^{-1} =$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{1}{A+1}\frac{d_1}{d_{k+2}} & 0 & \dots & 0 & \frac{-1}{A+1}\frac{d_1}{d_{2k+2}} \\ \frac{d_2}{d_1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{d_{k+1}}{d_k} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{A+1}\frac{d_{k+2}}{d_1} & 0 & \dots & 0 & \frac{-1}{A+1}\frac{d_{k+2}}{d_{k+1}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{d_{2k+2}}{d_{2k+1}} & 0 \end{pmatrix}$$

Now, we want to show that the sum of the absolute value of entries of each row is less than one, in order to find the infinite norm of DJD^{-1}. Since

$\varepsilon > 0$ so $1 - m\varepsilon > 1 - (m+1)\varepsilon$, that is, $d_m > d_{m+1}$, for all m. So

$$\frac{d_2}{d_1} < 1, \ \frac{d_3}{d_2} < 1, \ \ldots, \ \frac{d_{2k+2}}{d_{2k+1}} < 1$$

For
$$\begin{aligned}\frac{1}{A+1}\frac{d_{k+2}}{d_1} + \frac{1}{A+1}\frac{d_{k+2}}{d_{k+1}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\ &< \frac{1}{1-(k+1)\varepsilon}\frac{1}{(A+1)} + \frac{1}{1-(k+1)\varepsilon}\frac{1}{(A+1)} \\ &= \frac{2}{(1-(k+1)\varepsilon)(A+1)} \qquad \text{use Lemma 2.3.4} \\ &< 1\end{aligned}$$

Since J has the same eigenvalue as DJD^{-1}. Then,

$$\rho(J) = \max\{|\lambda_i|\} \leq \|DJD^{-1}\|_\infty$$

but

$$\|DJD^{-1}\|_\infty = \max\left\{ \tfrac{d_2}{d_1}, \tfrac{d_3}{d_2}, \ldots, \tfrac{d_{k+1}}{d_k}, \tfrac{1}{A+1} + \tfrac{1}{(1-(k+1)\varepsilon)(A+1)} \right\} < 1$$

So the modulus of every eigenvalue of J is less than one. Hence, the unique equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ of system (3.1) is locally asymptotically stable. □

Theorem 3.4.5. *If $A > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ of system (3.1) is globally asymptotically stable.*

Proof. Using previous theorem,the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ of system (3.1) is locally asymptotically stable. And by Theorem(2.3.2) the equilibrium point is global attractor, so the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ of system (3.1) is globally asymptotically stable. □

Theorem 3.4.6. *If $A \geq 1$, system (3.1) has no non-oscillatory positive solutions (has no infinite positive semi-cycle).*

Proof. On the contrary, assume system (3.1) has a non-oscillatory solution say $\{x_n, y_n\}_{n=-k}^{\infty}$ which has an infinite positive semi-cycle, and assume this semi-cycle starts with (x_N, y_N), where $N \geq -k$. Then for all $n \geq N$, $(x_n, y_n) \geq (A+1, A+1)$, hence

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} \geq A + 1 \quad \text{implies} \quad y_n \geq y_{n-k} \ \text{ for } \ n \geq N-1$$

and

$$y_{n+1} = A + \frac{x_n}{x_{n-k}} \geq A + 1 \quad \text{implies} \quad x_n \geq x_{n-k} \ \text{ for } \ n \geq N-1$$

using Theorem 3.4.2, there exist two real numbers Q, P such that for all $n \geq k+2$, $x_n \leq Q, y_n \leq P$. So for all $n \geq \max\{N-1, 2k+2\}$ (since we need $n - k \geq k+2$) we get

$$A + 1 \leq x_{n-k} \leq x_n \leq x_{n+k} \leq \cdots \leq Q \ \text{ and}$$
$$A + 1 \leq y_{n-k} \leq y_n \leq y_{n+k} \leq \cdots \leq P$$

similarly, in case $A = 1$, using Theorem 3.3.1, the solution is also bounded for all $n \geq -k$ which means that $\{x_n\}, \{y_n\}$ have k subsequences

$$\{x_{nk}\}, \{x_{nk+1}\}, ..., \{x_{nk+(k-1)}\} \ \text{ and } \ \{y_{nk}\}, \{y_{nk+1}\}, ..., \{y_{nk+(k-1)}\}$$

each subsequence is increasing and bounded from above, so each one of them is convergent, so for all $i = 0, 1, ..., k-1$ there exist γ_i, δ_i such that

$$\lim_{n\to\infty} x_{nk+i} = \gamma_i \text{ and } \lim_{n\to\infty} y_{nk+i} = \delta_i$$

Thus,

$$(\gamma_0, \delta_0), (\gamma_1, \delta_1), \ldots, (\gamma_{k-1}, \delta_{k-1})$$

is a k-periodic solution of system (3.1), which contradicts Theorem 3.1.2 unless the solution is the trivial solution. Hence, the solution converges to the equilibrium, which is a contradiction, because the solution is diverging from the equilibrium. Hence, system (3.1) has no non-oscillatory positive solutions when $A > 1$. □

Corollary 3.4.7. *If $A \geq 1$, then every solution $\{x_n, y_n\}_{n=-k}^{\infty}$ to system (3.1) oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ with semi-cycles such that if there exists a semi-cycle with at least k terms, then every semi-cycle after that has at least $k+1$ terms.*

Proof. Using Theorem 3.4.6, system (3.1) has no non-oscillatory positive solutions since $A \geq 1$, and Theorem 3.1.4 implies that system (3.1) has no non-oscillatory negative solutions. Thus, by Theorem 3.1.1 every solution of system (3.1) oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ with semi-cycles such that if there exists a semi-cycle with at least k terms, then every semi-cycle after that has at least $k+1$ terms. . □

3.5 Numerical Examples

In this section, we provide numerical examples done using MATLAB, to illustrate the results we have in chapter 3. Each example represent a different type of the dynamical behavior of solutions of (3.1).

Example 3.5.1. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-5}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-5}}, \quad n = 0, 1, \ldots \tag{3.23}$$

with $A = 0.5$, and the initial conditions $x_{-5} = 2.5$, $x_{-4} = 0.7$, $x_{-3} = 3$, $x_{-2} = 0.3$, $x_{-1} = 3.5$, $x_0 = 0.2$, $y_{-5} = 0.2$, $y_{-4} = 3.3$, $y_{-3} = 0.2$, $y_{-2} = 2.3$, $y_{-1} = 0.7$, $y_0 = 3.9$. Then the solution of system (3.23) is unbounded because $0 < A < 1$ (see Theorem 3.2.1), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (1.5, 1.5)$ is not globally asymptotically stable (see Figure 3.1).

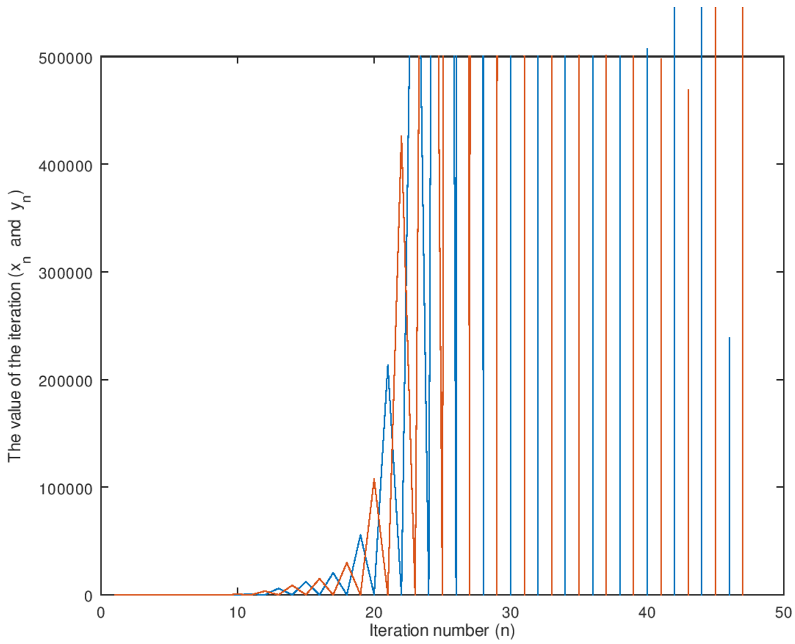

Figure 3.1: The plot of the positive solution of system (3.23) with $A = 0.5$

Example 3.5.2. Consider system (3.23) with $A = 2$, and the initial conditions $x_{-5} = 1.5,\ x_{-4} = 0.3,\ x_{-3} = 2,\ x_{-2} = 0.3,\ x_{-1} = 2.5,\ x_0 = 4,\ y_{-5} = 0.2,\ y_{-4} = 3,\ y_{-3} = 1.2,\ y_{-2} = 2.1,\ y_{-1} = 1.8,\ y_0 = 0.9$. Since $A > 1$, the solution of system (3.23) is bounded and persists (see Theorem 3.4.2), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (3, 3)$ is globally asymptotically stable (see Figure 3.2, Theorem 3.4.5).

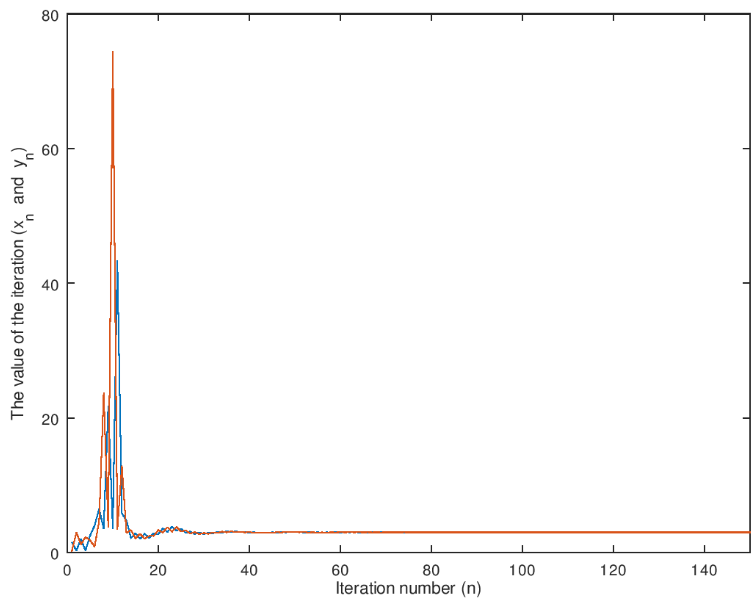

Figure 3.2: The plot of the positive solution of system (3.23) with $A = 2$

Example 3.5.3. Consider the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-6}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-6}}, \quad n = 0, 1, \ldots \tag{3.24}$$

with $A = 3$, and the initial conditions $x_{-6} = 3.6,\ x_{-5} = 0.1,\ x_{-4} = 1.1,\ x_{-3} = 4.3,\ x_{-2} = 0.5,\ x_{-1} = 0.2,\ x_0 = 1.1,\ y_{-6} = 2,\ y_{-5} = 0.9,\ y_{-4} = 2.2,\ y_{-3} = 1.5,\ y_{-2} = 0.7,\ y_{-1} = 3.9,\ y_0 = 3$. Then the unique positive equilibrium $(\bar{x}, \bar{y}) = (4, 4)$ is globally asymptotically stable since $A > 1$ (see Theorem 3.4.5), and the solution of system (3.24) is bounded and persists (see Figure 3.3, Theorem 3.4.2). Note that in this example $k = 6$ is even, while in Example 3.5.2, $k = 5$ is odd, but in both cases we had the same conclusion

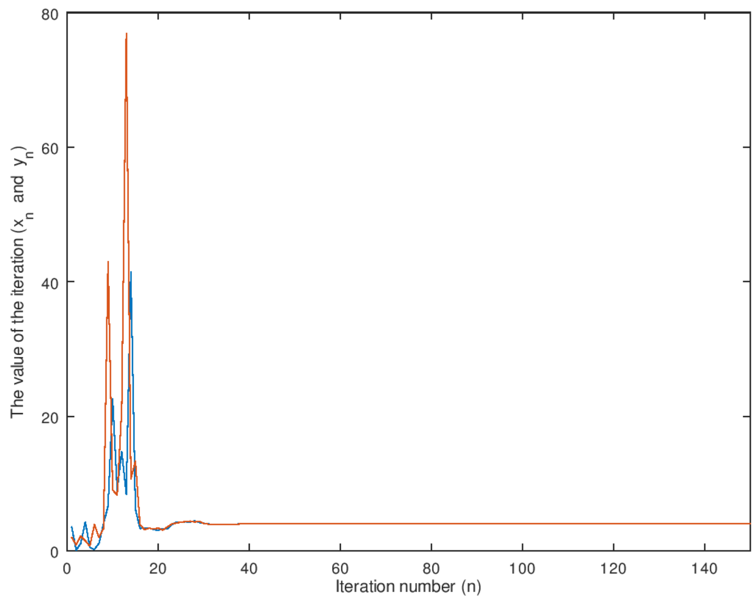

Figure 3.3: The plot of the positive solution of system (3.24) with $A = 3$

Example 3.5.4. Consider system (3.23) with $A = 1$, and the initial conditions $x_{-5} = 3$, $x_{-4} = 1.1$, $x_{-3} = 2.2$, $x_{-2} = 1.5$, $x_{-1} = 3$, $x_0 = 1.5$, $y_{-5} = 0.5$, $y_{-4} = 3$, $y_{-3} = 1.5$, $y_{-2} = 4$, $y_{-1} = 1$, $y_0 = 3$. Since $A = 1$ and $k = 5$ is an odd integer, then the solution of system (3.23) is two periodic solution (see Theorem 3.3.3), and the solution is also bounded (see Theorem 3.3.1), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (2, 2)$ is not globally asymptotically stable (see Figure 3.4).

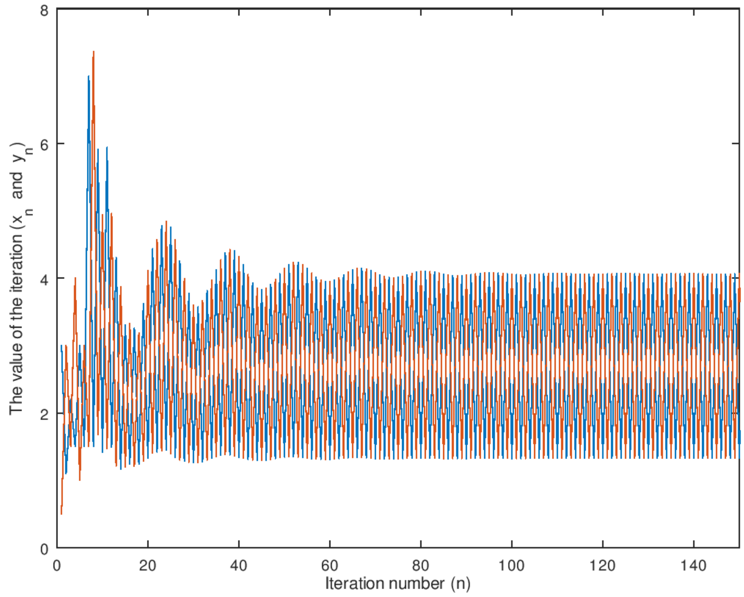

Figure 3.4: The plot of the positive solution of system (3.23) with $A = 1$

Example 3.5.5. Consider the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-4}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-4}}, \quad n = 0, 1, \ldots \tag{3.25}$$

with $A = 1$, and the initial conditions $x_{-4} = 3,\ x_{-3} = 1.5,\ x_{-2} = 3,\ x_{-1} = 1.5,\ x_0 = 3,\ y_{-4} = 1.5,\ y_{-3} = 3,\ y_{-2} = 1.5,\ y_{-1} = 3,\ y_0 = 1.5$. Since $A = 1$ and $k = 4$ is an even integer, Then the only two periodic solution of (3.25) is the equilibrium solution (see Theorem 3.3.3), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (2, 2)$ is globally asymptotically stable (see Figure 3.5).

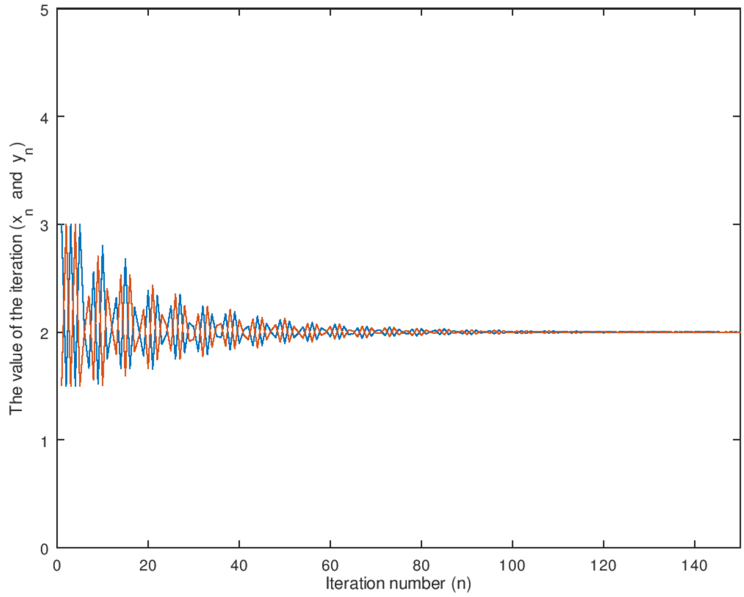

Figure 3.5: The plot of the positive solution of system (3.25) with $A = 1$

Conclusion

In this research, we solved an open problem proposed in [10] by Gumus (2018). We expanded the work on system (1.9) to a system with different parameters and investigated its dynamical behavior. We also introduced the symmetrical system of two rational difference equations (3.1) and studied the global behavior of its positive solutions.

Future Work

Our research can be expanded into more complicated related systems. The study of systems (3.1), (1.1) and (1.6) can be extended to systems with distinct parameters. Whereas system (2.1) can be extended to a system with different parameters and powers. Now, we will give some open problems that can be investigated next.

Open Problem 1. Investigate the dynamical behavior of the system

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \ldots$$

with parameters $A, B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$.

Open Problem 2. Investigate the dynamical behavior of the system of two difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{y_{n-k}}, \quad n = 0, 1, \ldots$$

where the parameters A, B are positive, the initial conditions $x_i, y_i \in (0, \infty)$ for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$.

Open Problem 3. Investigate the dynamical behavior of the system

$$x_{n+1} = A + \frac{y_{n-k}}{x_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{y_n}, \quad n = 0, 1, \ldots$$

with parameters $A, B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$.

Open Problem 4. Investigate the dynamical behavior of the system of two nonlinear difference equations

$$x_{n+1} = A + \frac{y_{n-k}^p}{y_n^q}, \quad y_{n+1} = B + \frac{x_{n-k}^p}{x_n^q}, \quad n = 0, 1, \ldots$$

where the parameters $A, B > 0$, the parameters p, q are nonnegative, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, ..., 0$ and $k \in \mathbb{Z}^+$.

Bibliography

[1] Abu-Saris, R. M., Devault, R. (2003). Global Stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. *Applied Mathematics Letters*, *16*(2), $173 - 178$.

[2] Camouzis, E., Papaschinopoulos, G. (2004). Global asymptotic behavior of positive solutions on the system of rational difference equations $x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, y_{n+1} = 1 + \frac{y_n}{x_{n-m}}$, *Applied Mathematics Letters*, *17*(6), $733 - 737$.

[3] Clark, D., Kulenovic, M. R. S., Selgrade, J. F. (2003). Global asymptotic behavior of a two-dimensional difference equation modelling competition. *Nonlinear Analysis*, *52*, $1765 - 1776$.

[4] Devault, R., Kent, C., and Kosmala, W. (2003). On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$. *Journal of Difference Equations and Applications*, *9*(8), $721 - 730$.

[5] Din, Q. (2014). On the system of rational difference equations. *Demonstratio Mathematica*, *47*(2), $324 - 335$.

[6] Din, Q., Ibrahim, T. F., and Khan, K. A. (2014). Behavior of a competitive system of second order difference equations. *The Scientific World Journal*, *2014*, Article ID: 283982.

[7] Elaydi, S. (1996). *An introduction to difference equations.* New York, NY: Springer-Verlag New York.

[8] Elaydi, S. (2007). *Discrete chaos: with applications in science and engineering.* Boka Raton, FL: Chapman and Hall/CRC.

[9] El-Owaidy, H. M., Ahmed, A. M., and Mousa, M. S. (2004). On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$. *Applied Mathematics and Computation*, *147*(1), 163 – 167.

[10] Gumus, M. (2018). The global asymptotic stability of a system of difference equations. *Journal of Difference Equations and Applications*, *24*(6), 976 – 991.

[11] Hu, G. (2016). Global behavior of a system of two nonlinear difference equations. *World Journal of Research and Review*, *2*(6), 36 – 38.

[12] Okumus, I., Soykan, Y. (2018). Some technique to show the boundedness of rational difference equations. *Journal of Progressive Research in Mathematics*, *13*(2), 2246 – 2258.

[13] Ozban, A. Y. (2006). On the positive solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$. *Mathematical Analysis and Applications*, *323*, 26 – 32.

[14] Ozban, A. Y. (2007). On the system of rational difference equations $x_n = \frac{a}{y_{n-3}}, y_n = \frac{by_{n-3}}{x_{n-q}y_{n-q}}$. *Applied Mathematics and Computation*, *188*, $833-837$.

[15] Papaschinopoulos, G., Papadopoulos, B. K. (2002). On the fuzzy difference equation $x_{n+1} = A + \frac{x_n}{x_{n-m}}$. *Fuzzy Sets and Systems*, *129*, $73-81$.

[16] Papaschinopoulos, G., Schinas, C. J. (1998). On a system of two nonlinear difference equations. *Journal of Mathematical Analysis and Applications*, *219*(2), $415-426$.

[17] Papaschinopoulos, G., Schinas, C. J. (2000). On the system of two difference equations $x_{n+1} = A + \frac{x_{n-1}}{y_n}, y_{n+1} = A + \frac{y_{n-1}}{x_n}$. *International Journal of Mathematics and Mathematical Science*, *23*(12), $839-848$.

[18] Saleh, M., Aloqeili, M. (2005). On the rational difference equation $y_{n+1} = A + \frac{y_{n-k}}{y_n}$. *Applied Mathematics and Computation*, *171*, $862-869$.

[19] Saleh, M., Aloqeili, M. (2006). On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$. *Applied Mathematics and Computation*, *176*, $359-363$.

[20] Saleh, M., Aloqeili, M. (2006). On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. *Applied Mathematics and Computation*, *177*, $189-193$.

[21] Yang, X., Liu, Y., Bai, S. (2005). On the system of high order rational difference equations $x_n = \frac{a}{y_{n-p}}, y_n = \frac{by_{n-p}}{x_{n-q}y_{n-p}}$. *Applied Mathematics and Computation*, *171*, $853-856$.

[22] Zhang, D., Ji, W., Wang, L., and Li, X. (2013). On the symmetrical system of rational difference equations $x_{n+1} = A + \frac{y_{n-k}}{y_n}, y_{n+1} = A + \frac{x_{n-k}}{x_n}$. *Appl. Math.*, *4*, $834-837$.

[23] Zhang, Q., Zhang, W., Shao, Y., and Liu, J. (2014). On the System of High Order Rational Difference Equations. *International Scholarly Research Notices*, *2014*, $1-5$.

[24] Zhang, Q., Yang, L., and Liu, J. (2013). On the Recursive System $x_{n+1} = A + \frac{x_{n-m}}{y_n}, y_{n+1} = A + \frac{y_{n-m}}{x_n}$. *Acta Math. Univ. Comenianae*, *82*(2), $201-208$.

[25] Zhang, Y., Yang, X. F., Evans, D. J., and Zhu, C. (2007). On the Nonlinear Difference Equation System $x_{n+1} = A + \frac{y_{n-m}}{x_n}, y_{n+1} = A + \frac{x_{n-m}}{y_n}$. *Computers and Mathematics with Applications*, *53*(10), $1561-1566$.